

Spin Wavelets on the Sphere

Daryl Geller

Department of Mathematics, University of Stony Brook
`daryl@math.sunysb.edu`

Domenico Marinucci

Department of Mathematics, University of Rome Tor Vergata
`marinucc@axp.mat.uniroma2.it`

December 15, 2008

Abstract

In recent years, a rapidly growing literature has focussed on the construction of wavelet systems to analyze functions defined on the sphere. Our purpose in this paper is to generalize these constructions to situations where sections of line bundles, rather than ordinary scalar-valued functions, are considered. In particular, we propose *needlet-type spin wavelets* as an extension of the needlet approach recently introduced by [37, 38] and then considered for more general manifolds by [18, 19, 20]. We discuss localization properties in the real and harmonic domains, and investigate stochastic properties for the analysis of spin random fields. Our results are strongly motivated by cosmological applications, in particular in connection to the analysis of Cosmic Microwave Background polarization data.

Keywords and phrases: *Wavelets, Frames, Line Bundles, Sphere, Spherical Harmonics, Spin, Needlets, Spin Needlets, High Frequency Asymptotics, Cosmic Microwave Background radiation, Polarization*
AMS Classification: 42C40, 60G60, 33C55, 14C21, 83F05, 58J05

1 Introduction

In recent years a rapidly growing literature has focussed on the construction of wavelet systems on the sphere, see for instance [3], [51] [1, 2] and the references therein. These attempts have been motivated by strong interest from the applied sciences, for instance in the areas of Geophysics, Medical Imaging and especially Cosmology/Astrophysics.

As far as the latter are concerned, special emphasis has been devoted to wavelet techniques for the statistical study of the Cosmic Microwave Background (CMB) radiation data. These data can be viewed as providing observations on the Universe in the immediate sequel to the Big Bang, and as such they have been the object of immense theoretical and applied interest over the last decade [12]. In particular, experiments such as the WMAP satellite from NASA have provided high resolution observations on the "temperature" (i.e. intensity) of CMB radiation from all directions of the full-sky ([27]). These observations have allowed precise estimates of several parameters of the greatest interest for Cosmology and Theoretical Physics. Spherical wavelets have found very extensive applications here, see for instance ([23, 9, 46, 8, 35, 36, 52]) and many others. The rationale for such a widespread interest can be explained as follows: CMB models are best analyzed in the frequency domain, where the behavior at different multipoles can be investigated separately; on the other hand, partial sky coverage and other missing observations make the evaluation of exact spherical harmonic transforms troublesome. The combination of these two features makes the time-frequency localization properties of wavelets most valuable.

Besides providing measurements on the radiation intensity, experiments such as WMAP have also provided some preliminary observations of a much more elusive physical entity, the so-called polarization of the background radiation. Polarization is a property of electromagnetic radiation whose physical significance is described for instance in ([28, 53]), see below for more mathematical discussion. So far,

empirical analysis of polarization has been somewhat limited, because this signal is currently measured with great difficulty. The situation with respect to CMB polarization data, however, will significantly improve over the next years, for instance by means of the ESA satellite mission Planck (expected to be launched in Spring 2009) which will take full-sky measurements of the polarized CMB sky ([31]) with much greater precision. Moreover, both ESA and NASA are planning high sensitivity full-sky satellite borne experiments within the next 10-20 years.

Polarization measurements are of extreme interest to physicists for several reasons. Indeed, not only do they allow improved precision for estimates of physical parameters which are already the focus of CMB temperature data, but they also open entirely new areas of research. Just to mention a striking example, at large scales the polarization signal is expected to be dominated by a component related to a background of gravitational waves which originated in the Big Bang dynamics (in the so-called inflationary scenario, see ([12])). Detection of this signal would be an outstanding empirical validation of many Theoretical Cosmology claims, directly related to Big Bang dynamic models but deeply rooted in General Relativity. Taking into account the huge amount of polarization data which will be available in the next 1-2 decades, as well as the important cosmological information contained in these data, it is clear that suitable mathematical tools for data analysis are in high demand. While a large amount of mathematical statistics techniques have been developed for analyzing CMB temperature data, far fewer mathematical tools are available to analyze polarization data.

From the mathematical point of view, as we shall detail below, polarization can be viewed as a (random) section of a line bundle on the sphere. Our purpose here is then to extend spherical wavelet constructions to the case where sections of line bundles, rather than ordinary functions, are considered. To the best of our knowledge, this is the first attempt to introduce wavelet techniques for the case where one deals with more general mathematical structures than ordinary (scalar-valued) functions on manifolds. We believe this area can be expanded into disciplines other than cosmology - for instance, tensor-valued random fields emerge naturally in brain imaging data ([43]).

In particular, in this paper we shall extend to the line bundle case the so-called needlet approach to spherical wavelets. Spherical needlets were recently introduced by [37, 38], and further developed, and extended to general smooth compact Riemannian manifolds in [18, 19, 20]. In a random fields environment, needlets were investigated by ([4, 5]), with a view to applications to the statistical analysis of CMB data; applications in the physical literature include [41], [33], [25], see also ([29], [11, 14, 15]) and [30, 34]. More precisely, we will focus on *needlet-type spin wavelets*, and we will argue below that they enjoy properties analogous to those of the usual needlets in the standard scalar case. In particular, we shall show below that needlet-type spin wavelets enjoy both the localization and the uncorrelation properties that make scalar needlets a powerful tool for the analysis of scalar-valued spherical random fields. More details on the plan and significance of our paper are given in the next section.

The companion article [17], intended primarily for physicists, goes into further detail about the statistical uses of spin wavelets. One should be aware, however, that there are some notational differences between that article and this one.

2 Plan of the Paper and Significance of the Results

The spin s concept that we will use in this paper originates in the fundamental work of Newman and Penrose [39]. Writing for physicists, they say that a quantity η defined on the sphere has spin weight $s \in \mathbb{Z}$, provided that, whenever a tangent vector m at a point transforms under coordinate change by $m' = e^{i\psi} m$, the quantity η , at that point, transforms by $\eta' = e^{is\psi} \eta$. They then develop a theory of spin s spherical harmonics, upon which we shall build to produce spin wavelets.

In section 3, we put the notion of spin s into acceptable mathematical language. We do this in an elementary manner which remains very close to the spirit of the work of Newman and Penrose. For another approach, using much more machinery, see [13].

Let \mathbf{N} be the north pole $(0, 0, 1)$, let \mathbf{S} be the south pole, $(0, 0, -1)$, and let U_I be $S^2 \setminus \{\mathbf{N}, \mathbf{S}\}$. If

$R \in SO(3)$, we define $U_R = RU_I$. On U_I we use standard spherical coordinates (θ, ϕ) ($0 < \theta < \pi$, $-\pi \leq \phi < \pi$), and analogously, on any U_R we use coordinates (θ_R, ϕ_R) obtained by rotation of the coordinate system on U_I . At each point p of U_R we let $\rho_R(p)$ be the unit tangent vector at p which is tangent to the circle $\theta_R = \text{constant}$, pointing in the direction of increasing ϕ_R . (Thus, for any $p \in U_I$, $\rho_R(Rp) = R_{*p}[\rho_I(p)]$.) If $p \in U_{R_1} \cap U_{R_2}$, we let $\psi_{pR_2R_1}$ be the angle from $\rho_{R_1}(p)$ to $\rho_{R_2}(p)$.

Now say $\Omega \subseteq S^2$ is open. We say that $f = (f_R)_{R \in SO(3)} \in C_s^\infty(\Omega)$, or that f is a smooth spin s function on Ω , provided all $f_R \in C^\infty(U_R \cap \Omega)$, and that for all $R_1, R_2 \in SO(3)$ and all $p \in U_{R_1} \cap U_{R_2} \cap \Omega$,

$$f_{R_2}(p) = e^{is\psi} f_{R_1}(p), \quad (1)$$

where $\psi = \psi_{pR_2R_1}$. If, say, $R_1 = I$, $R_2 = R$, then heuristically f_R is f_I “looked at after the coordinates have been rotated by R "; at p , it has been multiplied by $e^{is\psi}$. The angle ψ would be clearly be the same if we had instead chosen $\rho_R(p)$ to point in the direction of increasing θ_R , say, so this is an appropriate way to make Newman and Penrose's concept precise. Note that if $s = 0$, we can clearly identify $C_s^\infty(\Omega)$ with $C^\infty(\Omega)$.

For any s , we can identify $C_s^\infty(\Omega)$ with the sections over Ω of the complex line bundle obtained by using the $e^{is\psi_{pR_2R_1}}$ as transition functions from the chart U_{R_1} to the chart U_{R_2} ; then f_R is the trivialization of the section over $U_R \cap \Omega$. This is the point of view that we take in section 3, since, as is often the case in mathematics, certain properties are clearer, and more easily verified, if one uses the coordinate-free line bundle point of view. However, right now, for the reader's convenience, we state our results without reference to line bundles. Line bundles are rarely mentioned explicitly in this article after section 3. We may define $L_s^2(\Omega)$ (resp. $C_s(\Omega)$), by requiring that the f_R be in $L^2(U_R \cap \Omega)$ (resp. $C(U_R \cap \Omega)$) instead of $C^\infty(U_R \cap \Omega)$. There is a well-defined inner product on $L_s^2(\Omega)$, given by $\langle f, g \rangle = \langle f_R, g_R \rangle$; clearly this definition is independent of choice of R . A key observation of section 3 is that there is a unitary action of $SO(3)$ on $L_s^2(S^2)$, given by $f \rightarrow f^R$, which is determined by the equation $(f^R)_I(p) = f_R(Rp)$. We think of f^R as a “rotate” of f .

In section 4, we explain the spin s theory of Newman and Penrose, within the rigorous framework we have just outlined. Most of our arguments are very close to theirs.

For f smooth as above, following Newman and Penrose we define

$$\partial_{sR} f_R = -(\sin \theta_R)^s \left(\frac{\partial}{\partial \theta_R} + \frac{i}{\sin \theta_R} \frac{\partial}{\partial \phi_R} \right) (\sin \theta_R)^{-s} f_R,$$

and we show that the “spin-raising” operator $\partial : C_s^\infty(\Omega) \rightarrow C_{s+1}^\infty(\Omega)$ given by $(\partial f)_R = \partial_{sR} f_R$ is well-defined. We also define $\bar{\partial}_{sR} f_R$ by $\bar{\partial}_{sR} f_R = \overline{\partial_{sR} f_R}$, which leads to the spin-lowering operator $\bar{\partial} : C_s^\infty(\Omega) \rightarrow C_{s-1}^\infty(\Omega)$ given by $(\bar{\partial} f)_R = \bar{\partial}_{sR} f_R$. We show that ∂ and $\bar{\partial}$ commute with the actions of $SO(3)$ on smooth spin functions.

Now, for $l \geq |s|$, let $b_{ls} = [(l+s)!/(l-s)!]^{1/2}$ for $s \geq 0$, $b_{ls} = [(l-s)!/(l+s)!]^{1/2}$ for $s < 0$. Let $\{Y_{lm} : l \geq 0, -l \leq m \leq l\}$ be the standard basis of spherical harmonics on S^2 . Following Newman and Penrose, for $l \geq |s|$, we define the spin s spherical harmonics by ${}_s Y_{lm} = \partial^s Y_{lm}/b_{ls}$ for $s \geq 0$, ${}_s Y_{lm} = (-\bar{\partial})^s Y_{lm}/b_{ls}$ for $s < 0$, and we show that $\{{}_s Y_{lm} : l \geq |s|, -l \leq m \leq l\}$ forms an orthonormal basis for $L_s^2(S^2)$. There is a (relatively) simple explicit expression for the ${}_s Y_{lmI}$; it has the form

$${}_s Y_{lmI}(\theta, \phi) = {}_s y_{lm}(\theta) e^{im\phi} \quad (2)$$

for a suitable function ${}_s y_{lm}$. (See (60) for the explicit formula.) If $R \neq I$, ${}_s Y_{lmR}$ does not have a simple expression; but ${}_s Y_{lmR}^R(\theta_R, \phi_R)$ equals ${}_s y_{lm}(\theta_R) e^{im\phi_R}$ in the coordinates on U_R . (Here again ${}_s Y_{lm}^R$ denotes the “rotate” of ${}_s Y_{lm}$.)

There is also an analogue Δ_s of the spherical Laplacian for spin s functions; specifically, if $s \geq 0$, we let $\Delta_s = -\bar{\partial}\partial$, while if $s < 0$, we let $\Delta_s = -\partial\bar{\partial}$. Then Δ_0 is the usual spherical Laplacian. For $l \geq |s|$, let $\mathcal{H}_{ls} = \{ {}_s Y_{lm} : -l \leq m \leq l \}$. Also, for $l \geq |s|$, let $\lambda_{ls} = (l-s)(l+s+1)$ if $s \geq 0$, and let $\lambda_{ls} = (l+s)(l-s+1)$ if $s < 0$. Then \mathcal{H}_{ls} is the subspace of $C_s^\infty(S^2)$ which consists of eigenfunctions of

Δ_s for the eigenvalue λ_{ls} . This of course generalizes the situation in which $s = 0$. All of the aforementioned facts follow by making the arguments of Newman and Penrose rigorous.

In sections 5, 6, 7 and 8, we present new results. In section 5, we define operators with smooth kernels, from $C_s^\infty(S^2)$ to itself. We say $K = (K_{R',R})_{R',R \in SO(3)}$ is such a smooth kernel if each $K_{R',R} \in C^\infty(S^2 \times S^2)$, and if $K_{R_1',R_1}(x,y) = e^{is(\psi'-\psi)} K_{R',R}(x,y)$ for all $x \in U_{R_1'} \cap U_{R'}$ and $y \in U_{R_1} \cap U_R$, where $\psi' = \psi_{xR_1'R'}$ and $\psi = \psi_{yRR_1}$. It is then evident that we may consistently define an operator $\mathcal{K} : C_s^\infty(S^2) \rightarrow C_s^\infty(S^2)$ by $(\mathcal{K}f)_{R'}(x) = \int_{S^2} K_{R',R}(x,y) f_R(y) dS(y)$, for all $x \in U_{R'}$; we call \mathcal{K} the operator with kernel K . (These operators may be identified with operators with smooth kernels acting on sections of the aforementioned line bundle.)

Next we show that there is a good notion of spin s *zonal harmonic*. To see how this arises, we first show (in section 3) that, if $f \in C_s(S^2)$, then the function $e^{is\phi} f_I(\theta, \phi)$ extends continuously from U_I to $U_I \cup \mathbf{N}$, and the function $e^{-is\phi} f_I(\theta, \phi)$ extends continuously from U_I to $U_I \cup \mathbf{S}$. (This is shown by considering f_R for those R with $\mathbf{N}, \mathbf{S} \in U_R$.) Thus there is a well defined linear functional $L : C_s(S^2) \rightarrow \mathbb{C}$, given by $Lf = \lim_{\theta \rightarrow 0^+} e^{is\phi} f_I(\theta, \phi)$. It is evident from (2) that $L(sY_{lm})$ must be zero if $m \neq -s$. This indicates that $_s Y_{l,-s}$ could have a special status; in fact, it is a multiple of the spin s zonal harmonic, in the sense of the following results, which we prove in section 5.

Lemmas on Zonal Harmonics Define $_s Z_l = (-1)^{s^+} \sqrt{(2l+1)/4\pi} {}_s Y_{l,-s}$. Then:

$$(a) L(_s Z_l) = (2l+1)/4\pi.$$

$$(b) \text{For any } f \in \mathcal{H}_{ls}, Lf = \langle f, {}_s Z_l \rangle.$$

Next, let P_{ls} be the projection in $L^2(S^2)$ onto \mathcal{H}_{ls} , so that P_{ls} has kernel K^{ls} , where for any rotations R, R' , $K_{R',R}^{ls}(x,y) = \sum_{m=-l}^l {}_s Y_{lmR'}(x) \overline{{}_s Y_{lmR}(y)}$ for $x \in U_{R'}$, $y \in U_R$. Then:

(c) If $x \in U_I \cap U_{R'}$ and $\mathbf{N} \in U_R$, then $_s Z_{lI}(x) = e^{is(\psi_1-\psi_2)} K_{R',R}^{ls}(x, \mathbf{N})$, where $\psi_1 = \psi_{xR'I}$, and ψ_2 is the angle from $\rho_R(\mathbf{N})$ to $\partial/\partial y$ at \mathbf{N} .

(d) If $x \in U_R$, then $K_{R,R}^{ls}(x,x) = (2l+1)/4\pi$.

In section 6, we begin our study of spin wavelets. Before explaining our construction, we need to explain the ideas which are used in the spin 0 case on the sphere, and on more general manifolds. The references here are [37], [38], [18], [19], [20].

A word about notation: in sections 3-5 we often use the variable “ f ” to denote a spin function on the sphere. In sections 6-8 the variable “ f ” will be reserved for another purpose.

Specifically, say $f \in \mathcal{S}(\mathbb{R}^+)$, $f \neq 0$, and $f(0) = 0$. Let (\mathbf{M}, g) be a smooth compact oriented Riemannian manifold, and let Δ be the Laplace-Beltrami operator on \mathbf{M} (for instance, the spherical Laplacian if \mathbf{M} is S^2). Let K_t be the kernel of $f(t^2\Delta)$. Then, as we shall explain, the functions

$$w_{t,x}(y) = \overline{K_t}(x, y), \quad (3)$$

if multiplied by appropriate weights, can be used as wavelets on \mathbf{M} . In case $\mathbf{M} = S^n$ and f has compact support away from the origin, we shall say these wavelets are “needlet-type”; the theory of needlets was developed in [37], [38]. (Actually, the definition of needlet in [37], [38] is slightly different from (104), as we shall explain.) The theory was worked out in full generality in [18], [19], [20].

Specifically, one starts with the *Calderón formula*: if $c \in (0, \infty)$ is defined by $c = \int_0^\infty |f(t)|^2 \frac{dt}{t}$, then for all $u > 0$,

$$\int_0^\infty |f(tu)|^2 \frac{dt}{t} = c < \infty. \quad (4)$$

Discretizing (4), if $a > 1$ is sufficiently close to 1, one obtains a special form of *Daubechies’ condition*: for all $u > 0$,

$$0 < A_a \leq \sum_{j=-\infty}^{\infty} |f(a^{2j}u)|^2 \leq B_a < \infty, \quad (5)$$

where

$$A_a = \frac{c}{2|\log a|} (1 - O(|(a-1)^2(\log |a-1|)|)), \quad (6)$$

$$B_a = \frac{c}{2|\log a|} (1 + O(|(a-1)^2(\log |a-1|)|)). \quad (7)$$

((6) and (7) were proved in [16], Lemma 7.6. In particular, B_a/A_a converges nearly quadratically to 1 as $a \rightarrow 1$. For example, Daubechies calculated that if $f(u) = ue^{-u}$ and $a = 2^{1/3}$, then $B_a/A_a = 1.0000$ to four significant digits.

Let P be the projection in $L^2(S^2)$ onto the space of constant functions (the null space of Δ .) Suppose that $a > 1$ is sufficiently close to 1 that A_a and B_a are close (or that a, f have been chosen in such a manner that $B_a = A_a$). Then for an appropriate discrete set $\{x_{j,k}\}_{(j,k) \in \mathbb{Z} \times [1, N_j]}$ on \mathbf{M} , and certain weights $\mu_{j,k}$, the collection of $\{\phi_{j,k} := \mu_{j,k} w_{a^j, x_{j,k}}\}$ constitutes a wavelet frame for $(I - P)L^2(S^2)$. By this we mean that there exist ‘frame bounds’ $0 < A \leq B < \infty$, such that for any $F \in (I - P)L^2(S^2)$ the following holds:

$$A \| F \|_2^2 \leq \sum_{j,k} |\langle F, \phi_{j,k} \rangle|^2 \leq B \| F \|_2^2. \quad (8)$$

If the points $\{x_{j,k}\}$ are selected properly, B can be made arbitrarily close to B_a and A can be made arbitrarily close to A_a . Because of the proximity of A_a to B_a , the frame is therefore a ‘nearly tight frame’. In passing from one scale to another, we use ‘dilations’ in the sense that we are looking at the kernel of $f(t^2\Delta)$ for different t , which corresponds to dilating the metric on the manifold. On the sphere, the wavelets at each scale j are all rotates of each other, up to constant multiples. (This is analogous to the usual situation in which all the wavelets at each scale are translates of each other.)

Again, $w_{t,x}$ is given by (3). On the sphere S^2 , we have

$$K_t(x, y) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f(t^2 \lambda_l) Y_{lm}(x) \overline{Y_{lm}}(y). \quad (9)$$

The most important cases to consider on the sphere are the case in which f has compact support away from 0 (the ‘needlet-type’ case, of Narcowich, Petrushev and Ward), and the case in which $f(u) = u^r e^{-u}$ for some integer $r \geq 1$ (the ‘Mexican needlet’ case, as considered in [18], [19]). (Actually, in their work, Narcowich, Petrushev and Ward used l^2 in place of $l(l+1)$ in (105), but this is a minor distinction.) Needlet-type wavelets and Mexican needlets each have their own advantages. Needlet-type wavelets have these advantages: for appropriate f , a , $\mu_{j,k}$ and $x_{j,k}$, needlet-type wavelets are a tight frame on the sphere (i.e., $A = B$ in (8)), and the frame elements at non-adjacent scales are orthogonal. In fact, for appropriate needlet-type f , a , $\mu_{j,k}$ and $x_{j,k}$, one can arrange that $A_a = B_a = A = B = 1$ in the discussion above. In that case, the wavelets are called needlets.

Mexican needlets have their own advantages. In [18], [19], an approximate formula is written down for them which can be used directly on the sphere. (This formula, which arises from computation of a Maclaurin series, has been checked numerically. It is work in progress, expected to be completed soon, to estimate the remainder terms in this Maclaurin series. In [18], [19] the formula is written down only for $r = 1$, but it can be readily generalized to general r . The formula indicates that Mexican needlets are quite analogous to the Mexican hat wavelet which is commonly used on the real line.) Mexican needlets have Gaussian decay at each scale, and they do not oscillate (for small r). Thus they can be implemented directly on the sphere, which is desirable if there is missing data (such as the ‘sky cut’ of the CMB).

It would be worthwhile to utilize both needlets and Mexican needlets in the analysis of CMB, and the results should be compared. In this article we focus on needlet-type situations.

A key property of this kind of wavelet is its *localization*. Working in the general situation (general \mathbf{M} , general f), one has:

for every pair of C^∞ differential operators X (in x) and Y (in y) on \mathbf{M} , and for every nonnegative integer N , there exists $c := c_{N,X,Y}$ such that for all $t > 0$ and $x, y \in \mathbf{M}$

$$|XYK_t(x, y)| \leq c \frac{t^{-(n+I+J)}}{(d(x, y)/t)^N}, \quad (10)$$

where $I := \deg X$, $J := \deg Y$, and $d(x, y)$ denotes the geodesic distance from x to y . In the case where \mathbf{M} is the sphere, f has compact support away from 0, and if one replaces $\lambda_l = l(l+1)$ by l^2 in the formula for K_t , this was shown by Narcowich, Petrushev and Ward, in [37] and [38]. The general result was shown in [18].

The proof of (10) in [18] would in fact go through without change if, instead of K_t being the kernel of $f(t^2\Delta)$, where Δ is the Laplace-Beltrami operator, one assumed that K_t was the kernel of $f(t^2D)$, where D is any smooth positive second-order elliptic differential operator, acting on $C^\infty(\mathbf{M})$. It is natural to conjecture then that something analogous holds if D is instead a smooth positive second-order elliptic differential operator, between sections of line bundles on \mathbf{M} . Since Δ_s may be interpreted as being such an operator, it is natural to want to prove the following result.

Theorem 6.1

Let K_t be the kernel of $f(t^2\Delta_s)$. Then:

For every $R, R' \in SO(3)$, every pair of compact sets $\mathcal{F}_R \subseteq U_R$ and $\mathcal{F}_{R'} \subseteq U_{R'}$, and every pair of C^∞ differential operators X (in x) on $U_{R'}$ and Y (in y) on U_R , and for every nonnegative integer N , there exists c such that for all $t > 0$, all $x \in \mathcal{F}_{R'}$ and all $y \in \mathcal{F}_R$, we have

$$|XYK_{t,R',R}(x, y)| \leq c \frac{t^{-(n+I+J)}}{(d(x, y)/t)^N}, \quad (11)$$

where $I := \deg X$ and $J := \deg Y$.

In this article, we only give the proof of Theorem 6.1 in the easier ‘‘needlet-type’’ case, where f has compact support away from the origin. The general case will be established elsewhere.

Let us now explain what we mean by spin wavelets. For $R \in SO(3)$, and $x \in U_R$, let us define

$$w_{txR} = \sum_{l \geq |s|} \sum_m \overline{f}(t^2 \lambda_{ls}) \overline{sY_{lmR}}(x) sY_{lm}. \quad (12)$$

Then $w_{txR} \in C_s^\infty(S^2)$. Also note that $w_{txR,R'}(y) = \overline{K}_{t,R,R'}(x, y)$ (if $x \in U_R$ and $y \in U_{R'}$); this generalizes the case $s = 0$ of (3). Moreover, if $F \in L_s^2(S^2)$, then for $x \in U_R$

$$\beta_{t,F,x,R} := \langle F, w_{txR} \rangle = (f(t^2\Delta_s)F)_R(x) = (\beta_{t,F})_R(x) \quad (13)$$

where we have set $\beta_{t,F} = (f(t^2\Delta_s)F)$. We call $\beta_{t,F,x,R}$ a *spin wavelet coefficient* of F . If $F = \sum_{l \geq |s|} \sum_m a_{lm} sY_{lm} \in L_s^2(S^2)$, then

$$\beta_{t,F,x,R} = \sum_{l \geq |s|} \sum_m f(t^2 \lambda_{ls}) a_{lm} sY_{lmR}(x). \quad (14)$$

Let us now explain how, in analogy to the case $s = 0$, one can obtain a nearly tight frame from spin wavelets. Let $P = P_{|s|,s}$ be the projection onto the $\mathcal{H}_{|s|,s}$, the null space of Δ_{ls} (in C_s^∞). Suppose that $a > 1$ is sufficiently close to 1 that A_a and B_a (of (6) and (7)) are close (or that a, f have been chosen in such a manner that $B_a = A_a$). We then claim that for an appropriate discrete set $\{x_{j,k}\}_{(j,k) \in \mathbb{Z} \times [1, N_j]}$ on S^2 , certain $R_{j,k}$ with $x_{j,k} \in U_{R_{j,k}}$, and certain weights $\mu_{j,k}$, the collection of $\{\phi_{j,k} := \mu_{j,k} w_{a^j, x_{j,k}, R_{j,k}}\}$ constitutes a nearly tight wavelet frame for $(I - P)L_s^2(S^2)$. (Note that this property is independent

of the choice of $R_{j,k}$.) In that case, we will have that for some C , if $F \in (I - P)L_s^2$, then, in L^2 , $F \sim C \sum_{j,k} \langle F, \phi_{j,k} \rangle \phi_{j,k}$ (note that this sum is *independent* of the choice of $R_{j,k}$). The precise statement is Theorem 6.3 below, where we explain how the $x_{j,k}$ and $\mu_{j,k}$ are to be chosen. Theorem 6.3 will be proved elsewhere [21]; in this article, it is only used in section 8.

One can certainly choose a , and f with compact support away from the origin, so that $A_a = B_a = 1$ (recall (5)). Such f were used in [37] and [38]. It is then natural to conjecture, that as in the spin 0 case of [37] and [38], one can choose the points $x_{j,k}$ so that the $\{\mu_{j,k}\varphi_{j,k}\}$ are an exactly tight frame for $(I - P)L_s^2(S^2)$, for suitable weights $\mu_{j,k}$. One should note, however, that even if such explicit points and weights could be found, it might not be practical for physicists to use them in CMB analysis, where measurements are invariably taken at the HEALPix points [24]. This, then, is a situation where it is very useful (and perhaps essential) to have the flexibility in choice of the $x_{j,k}$ which Theorem 6.3 allows.

In section 7, we begin to examine how our theory can be applied to cosmology. We look at random spin s fields G on S^2 . We say that G is *isotropic* (in law) if for every $x_1, \dots, x_N \in U_I$, the joint probability distribution of $G_I^R(x_1), \dots, G_I^R(x_N)$ is independent of $R \in SO(3)$. Similarly, if G, H are both random spin s fields, we say that G, H are *jointly isotropic* (in law) if for every $x_1, \dots, x_N, y_1, \dots, y_M \in U_I$, the joint probability distribution of $G_I^R(x_1), \dots, G_I^R(x_N), G_I^R(y_1), \dots, G_I^R(y_M)$ is independent of $R \in SO(3)$. Say G, H are jointly isotropic, and denote their spin s spherical harmonic coefficients by $a_{lm} = \langle G^R, {}_s Y_{lm} \rangle$, $b_{lm} = \langle H^R, {}_s Y_{lm} \rangle$; these are random variables. In Theorem 7.2 we show that $E(a_{lm} \bar{b}_{l'm'}) = 0$ unless $l = l'$ and $m = m'$ (see [6] for the scalar case). Moreover $E(a_{lm} \bar{b}_{lm})$ does not depend on m ; we will denote it by $C_{l,G,H}$.

In Theorem 7.3, we generalize a key result in [4], to show that the spin wavelet coefficients of G, H are asymptotically uncorrelated, under mild hypotheses. That is, for $x \in U_R$, we define the spin wavelet coefficients $\beta_{G,t,x,R} = \langle G, w_{txR} \rangle$, $\beta_{H,t,x,R} = \langle H, w_{txR} \rangle$, and we show that $|\text{Cor}(\beta_{G,t,x,R}, \beta_{H,t,y,R})| \rightarrow 0$ as $t \rightarrow 0^+$. (Again we assume that we are in the ‘‘needlet-type’’ situation, where f has compact support away from 0.)

Finally, in section 8, we offer a brief glimpse of the consequences of the considerations of section 7 for stochastic limits and cosmology.

Say $\epsilon > 0$. We use a nearly tight frame, as provided by Theorem 6.3. Specifically, in that theorem, we fix a real f supported in $[1/a^2, a^2]$ for which the Daubechies sum $\sum_{j=-\infty}^{\infty} f^2(a^{2j}u) = 1$ for all $u > 0$, so that $A_a = B_a = 1$. We then produce the $\{\phi_{j,k}\} = \{\phi_{j,k,\epsilon}\}$ with frame bounds $B = 1 + \epsilon$ and $A = 1 - \epsilon$. Say $G = \sum A_{lm} {}_s Y_{lm}$ is an isotropic random spin s field, and let $C_l = C_{l,G,G}$ (notation as in Theorem 7.3). For $j \in \mathbb{Z}$, we also let $\gamma_j = \sum_{a^{-2} \leq a^{2j} \lambda_{ls} \leq a^2} C_l(2l+1)$; this is a small quantity since $\sum_j \gamma_j \leq 3 \sum_l C_l(2l+1) = 3\text{Var}G$. (In all sums here, l satisfies the implicit restriction $l \geq |s|$.) We let $\beta_{jk} = \beta_{jke} = \langle G, \phi_{j,k,\epsilon} \rangle$. We focus on the quadratic statistics

$$\tilde{\Gamma}_j = \sum_k |\beta_{jk}|^2 = \sum_k |\beta_{jke}|^2, \quad \hat{\Gamma}_j = \sum_{l,m} f^2(a^{2j} \lambda_{ls}) |A_{lm}|^2 \quad (15)$$

When considering $\tilde{\Gamma}_j$, we can even let ϵ depend on j here. In Proposition 8.1, we show that

$$E(|\hat{\Gamma}_j - \tilde{\Gamma}_j|) \leq \epsilon \gamma_j \quad (16)$$

for all j .

Since we can let $\epsilon \ll 1$ depend on j , by Tchebychev’s inequality, the probability distribution of $\tilde{\Gamma}_j$ is a small perturbation of that of $\hat{\Gamma}_j$.

We conclude, then, by briefly discussing the latter. We assume now that G is *Gaussian*, by which we mean that $\{\Re G_I(x) : x \in U_I\} \cup \{\Im G_I(x) : x \in U_I\}$ is a Gaussian family. We shall also assume that $G = \sum A_{lm} {}_s Y_{lm}$ is *involutive*, by which we mean that $\overline{A}_{lm} = A_{l,-m}$ for all m (so that, in particular, A_{l0} is real). This assumption is very natural to make in cosmology. Indeed, as we shall explain in section 4, Newman and Penrose explained how to write any random spin s field F as $F = F_{\mathbf{E}} + iF_{\mathbf{M}}$, where the

“electric” and “magnetic” parts $F_{\mathbf{E}}$ and $F_{\mathbf{M}}$ are involutive. In cosmology one assumes that these fields are jointly isotropic.

On the other hand, the assumption of Gaussianity is common to a large part of the literature on random fields, and it is predominant in the cosmological area, both within physical and mathematical articles. Concerning the physical literature, Gaussianity is motivated by the widely dominating inflationary model for the Big Bang dynamics, which predicts the polarization random field is the outcome of Gaussian fluctuations of a quantum mechanical origin. From the mathematical point of view, Gaussianity is nearly a mandatory assumption in the *high-frequency asymptotics* framework we shall entertain here (compare [32]); indeed, as a consequence of the characterization of isotropic random fields given in ([6]), the Gaussianity of the spherical harmonic coefficients is equivalent to their stochastic independence, which in turns make limit theorems feasible. The extension to non-Gaussian circumstances is certainly a crucial issue for further research; we stress, however, that in the framework of high-resolution asymptotics no rigorous results are known in the literature in non-Gaussian circumstances, even in the standard case of scalar random fields.

Under these assumptions, in Proposition 8.2 we prove a central limit theorem for the $\widehat{\Gamma}_j$, namely that

$$\frac{\widehat{\Gamma}_j - E\widehat{\Gamma}_j}{\sqrt{Var\{\widehat{\Gamma}_j\}}} \rightarrow_d N(0, 1) \text{ as } j \rightarrow -\infty,$$

\rightarrow_d denoting as usual convergence in probability law, and $N(0, 1)$ denoting as usual the standard normal distribution.

Let us explain the significance of this result. As we explained earlier, our main motivating rationale is provided by the observations of a polarization random field F in cosmological satellite experiments. We shall then consider the realistic situation where a physical model is adopted to provide an expected value for the “angular power spectrum” C_l , of the random spin field F , typically as a function $C_l = C_l(\Psi)$, of some vector of physical parameters Ψ . In the case of cosmological polarization data, the function $C_l(\cdot)$ is estimated by the numerical solution of partial differential (thermodynamic) equations representing the evolution of a matter/energy fluid, and the parameters Ψ are related to constants of fundamental physics. Propositions 8.1 and 8.2 highlight the possibility of using spin quantities, such as the spin wavelet coefficients β_{jk} for F , to provide asymptotically valid estimators for scalar quantities of physical interest such as C_l . For instance, by Propositions 8.1 and 8.2, it is immediate to construct a test for the assumption that the angular power spectra take the form $C_l(\Psi)$, for some physically motivated set of parameters which takes a given value $\Psi = \Psi_0$. It is indeed sufficient to focus on

$$S_j := \frac{\widehat{\Gamma}_j - E_{\Psi_0}\widehat{\Gamma}_j}{\sqrt{Var_{\Psi_0}\{\widehat{\Gamma}_j\}}},$$

where $E_{\Psi_0}\widehat{\Gamma}_j$, $Var_{\Psi_0}\{\widehat{\Gamma}_j\}$ are the expected mean and variance evaluated under the assumption that the vector of parameters is set equal to Ψ_0 . Propositions 8.1 and 8.2 predict that, for $-j$ sufficiently large and ϵ sufficiently small (depending on j), the probability of exceeding any given threshold value will be nearly provided by

$$\Pr\{S_j \geq z_{\alpha/2}\} \sim \alpha,$$

where $0 < \alpha < 1$ and $z_{\alpha/2}$ is the well-known quantile of the standard Gaussian distribution $\Phi(\cdot)$, i.e. $z_{\alpha/2} := \Phi^{-1}(1 - \alpha/2)$. Values of S_j above $z_{\alpha/2}$ provide evidence at *significance level* α against the assumption that Ψ_0 represent the physical parameters in the model that generated the observations.

We stress that the direct use of $\widehat{\Gamma}_j$ is only feasible when observations from the whole sky are available. In cosmology, one is hampered by the presence of unobserved sky regions, for instance due to the foreground emission by the Milky Way and other astrophysical sources. The localization properties of needlet-type

spin wavelets in the real and harmonic domains are then clearly most valuable because, outside the masked regions, wavelet coefficients in the high-frequency limit are essentially unaffected by the missing observations. Moreover, the uncorrelation properties of needlet coefficients in the Gaussian case make a sound asymptotic statistical theory feasible, even in the presence of unobserved regions. Furthermore, these dual localization properties make possible the search for features and cosmological asymmetries, as done in the scalar case by [42]. We refer to [17] for more discussion and details. (The reader should be aware that in the notation of [17], the high-frequency limit corresponds to letting $j \rightarrow \infty$, not $-\infty$ as in the present article.)

3 Spin s line bundles

In the next section, we shall review the groundbreaking work of Newman and Penrose [39] on spin weight s . They were writing for physicists, so it is necessary to place their observations in a rigorous mathematical context. We do this in a manner which remains very close to the spirit of the work of Newman and Penrose, and uses very little machinery; for another approach, using more machinery, see [13]. We also contribute some new results, some of which will be crucial for our work.

Newman and Penrose work on a two-dimensional surface. Writing for physicists, they say that a quantity η has spin weight $s \in \mathbb{Z}$, provided that, whenever a tangent vector m at a point transforms under coordinate change by $m' = e^{i\psi}m$, the quantity η , at that point, transforms by $\eta' = e^{is\psi}\eta$. Our first task is to put this notion into acceptable mathematical language.

Let \mathbf{M} be an oriented Riemann surface. The complex structure on \mathbf{M} determines a unique metric (inner product on real tangent vectors at each point), up to conformal equivalence (that is, up to a constant multiple at each point). (Indeed, locally, such a metric is the pullback through a chart of the standard metric on \mathbb{C} ; different choices of charts lead to conformally equivalent metrics, since the Jacobian of a conformal mapping is always a constant multiple of an orthogonal transformation.) We always use a metric on \mathbf{M} in this conformal class (that is, a metric which agrees with one determined by the complex structure, up to a constant multiple at each point) without further comment. Let us call this conformal class the *intrinsic conformal class* on \mathbf{M} .

If $p \in \mathbf{M}$, as usual we let \mathbf{M}_p denote the real tangent space at p . Since \mathbf{M} is oriented, for each $v \in \mathbf{M}_p$ we may select $Jv \in \mathbf{M}_p$ at that point, such that (v, Jv) is an oriented orthogonal basis for \mathbf{M}_p . Surely Jv is unique, up to a positive multiple. Thus, if v, w are nonzero tangent vectors at a point, and g is our metric, we may very naturally define the *angle from v to w* to be the angle $\psi \in (-\pi, \pi]$ determined by the two properties:

- (1) $g(v, w) = r \cos \psi$ for some $r > 0$; and
- (2) $\operatorname{sgn} \psi = \operatorname{sgn} g(Jv, w)$.

Now, say we have an atlas \mathcal{U} on \mathbf{M} . Suppose that for each chart $U_\alpha \in \mathcal{U}$ we have a section ρ_α of TU_α (the real tangent bundle of U_α). (This section will give, at each point of U_α , a “reference direction”.) Set $\rho = \{\rho_\alpha\}_{U_\alpha \in \mathcal{U}}$. Say also $s \in \mathbb{Z}$. We then define \mathbf{L}^s , the spin s line bundle associated to $(\mathbf{M}, \mathcal{U}, \rho)$, as follows. Say U_α, U_β are charts in \mathcal{U} , and define, for each $p \in U_\alpha \cap U_\beta$, $\psi_{p\beta\alpha}$ to be the angle from $\rho_\alpha(p)$ to $\rho_\beta(p)$. We then set

$$\lambda_{\beta\alpha}(p) = e^{is\psi_{p\beta\alpha}}.$$

Then $\lambda_{\alpha\beta}\lambda_{\beta\alpha} = 1$ on $U_\alpha \cap U_\beta$, and if U_γ is a chart in \mathcal{U} as well, then $\lambda_{\alpha\gamma}\lambda_{\gamma\beta}\lambda_{\beta\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. By a standard argument (see, for instance, the argument on [26], pages 139-140), the $\lambda_{\beta\alpha}$ may be used as transition functions to define a complex (not necessarily holomorphic) line bundle \mathbf{L}^s ; this bundle is unique up to isomorphism. If π is the projection in \mathbf{L}^s onto \mathbf{M} , there are therefore diffeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}$, such that if $p \in U_\alpha \cap U_\beta$, and if

$$\Phi_\alpha(p, \eta) = (p, z),$$

then

$$\Phi_\beta(p, \eta) = (p, e^{is\psi_{p\beta\alpha}} z).$$

Let us say that these maps $\{\Phi_\alpha\}$ are *induced* by (\mathcal{U}, ρ) .

Note that, if Ω is an open subset of \mathbf{M} , we may naturally restrict \mathbf{L}^s to become a line bundle \mathbf{L}_Ω^s over Ω , by using the charts $\{U_\alpha \cap \Omega : U \in \mathcal{U}\}$, and the section $\rho'_\alpha := \rho_\alpha$ on $U_\alpha \cap \Omega$.

Suppose now that $\mathbf{L}_1^s, \mathbf{L}_2^s$ are determined, respectively, by $(\mathbf{M}^1, \mathcal{U}^1, \rho^1)$ and $(\mathbf{M}^2, \mathcal{U}^2, \rho^2)$. Say $F : \mathbf{M}^1 \rightarrow \mathbf{M}^2$ is holomorphic, and is a local diffeomorphism. Then F naturally gives rise to a smooth map $F_s : \mathbf{L}_1^s \rightarrow \mathbf{L}_2^s$ as follows. Say (\mathcal{U}^1, ρ^1) induces the maps $\{\Phi_\alpha^1\}$, and (\mathcal{U}^2, ρ^2) induces the maps $\{\Phi_\beta^2\}$. Say $p \in U_\alpha^1$, a chart in \mathcal{U}^1 , and $q = F(p) \in U_\beta^2$, a chart in \mathcal{U}^2 . The map F gives rise to a mapping $F_{*p} : \mathbf{M}_p^1 \rightarrow \mathbf{M}_q^2$. Let ψ be the angle from $F_{*p}[\rho_\alpha^1(p)]$ to $\rho_\beta^2(q)$. Say now $(p, \eta) \in \pi^{-1}(U_\alpha^1)$, and that

$$\Phi_\alpha^1(p, \eta) = (p, z).$$

Then we define $F_s(p, \eta) = (q, \zeta)$, where

$$\Phi_\beta^2(q, \zeta) = (q, e^{is\psi} z).$$

Since F_* preserves angles, it is easy to see that this definition is independent of choice of α, β . Further, if $I : \mathbf{M}_1 \rightarrow \mathbf{M}_1$ is the identity map, so is $I_s : \mathbf{L}_1^s \rightarrow \mathbf{L}_2^s$.

Moreover, say F is actually a biholomorphic map. Then, since F_* and $(F^{-1})_*$ preserve angles, one easily sees that $(F_s)^{-1} = (F^{-1})_s$. If f is a (continuous) section of \mathbf{L}_2^s , we may naturally define a section f^F of \mathbf{L}_1^s by

$$f^F = F_s^{-1} \circ f \circ F. \quad (17)$$

If $G : \mathbf{M}^0 \rightarrow \mathbf{M}^1$ is another biholomorphic map, we then have

$$(f^F)^G = f^{F \circ G}. \quad (18)$$

This is all quite abstract, but in fact we shall study spin s bundles only when the manifold is an open subset of the sphere S^2 or the complex plane \mathbb{C} . On S^2 , by choosing the atlas and ρ well, we can impose a very interesting and very explicit structure.

On S^2 , realized as $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, we let \mathbf{N} be the north pole $(0, 0, 1)$, and let \mathbf{S} be the south pole, $(0, 0, -1)$. We define the chart U_I to be $S^2 \setminus \{\mathbf{N}, \mathbf{S}\}$. (The choice of chart map from U_I to \mathbb{C} is irrelevant; one can take it to be a stereographic projection.) We obtain all other charts in our atlas by rotating U_I . Thus, if $R \in SO(3)$, we define $U_R = RU_I$. On U_I we often use standard spherical coordinates (θ, ϕ) , and at each point p of U_I we let $\rho_I(p)$ be the unit tangent vector at p which is tangent to the circle $\theta = \text{constant}$. (For definiteness, if this circle is $x = r \cos \phi$, $y = r \sin \phi$, $z = \text{constant}$, let us choose $\rho_I(p)$ to point in the direction in which ϕ increases, the “counterclockwise” direction.) On any other chart U_R , we choose

$$\rho_R(Rp) = R_{*p}[\rho_I(p)]. \quad (19)$$

for any $p \in U_0$.

On the chart U_R we can use coordinates (θ_R, ϕ_R) , obtained by rotation of the (θ, ϕ) coordinates. Thus, if the (θ, ϕ) coordinates of $p \in U_I$ are (θ_0, ϕ_0) , then the (θ_R, ϕ_R) coordinates of $Rp \in U_R$ are also (θ_0, ϕ_0) . Then, at any $q \in U_R$, $\rho_R(q)$ is the unit tangent vector to the circle $\phi_R = \text{constant}$, pointing in the direction of increasing ϕ_R . Again, the angle $\psi_{pR_2R_1}$ is the angle from $\rho_{R_1}(p)$ to $\rho_{R_2}(p)$. We would clearly obtain the same angle if $\rho_R(p)$ had been chosen as the unit vector pointing in the direction of increasing θ_R , or in the direction of any fixed linear combination of the unit vectors pointing in the directions of increasing ϕ_R and increasing θ_R . In this precise sense, $\psi_{pR_2R_1}$ effectively measures the angle by which the tangent plane at p is rotated if one uses the $(\theta_{R_2}, \phi_{R_2})$ coordinates instead of the $(\theta_{R_1}, \phi_{R_1})$ coordinates. This is why one should think of our choice of $\rho_R(p)$ only as a convenient “reference direction”.

On \mathbb{C} we will use only a single chart, \mathbb{C} itself, and if $p \in \mathbb{C}$, we let our reference direction $\rho(p)$ be $\partial/\partial y$. Since there is only one chart, this bundle is trivial. However, if F maps an open subset U of \mathbb{C} conformally onto another open subset U of \mathbb{C} , we can consider the map F_s , which, as we shall see, can be quite interesting.

Now, if f is a (continuous) section of \mathbf{L}^s over an open set $\Omega \subseteq S^2$, then we may write

$$\Phi_R(f(q)) = (q, f_R(q))$$

for all $q \in U_R \cap \Omega$, for a suitable function $f_R : U_R \cap \Omega \rightarrow \mathbb{C}$. Note, then, that if $p \in U_I \cap U_R \cap \Omega$, then

$$f_R(p) = e^{is\psi} f_I(p) \quad (20)$$

where

$$\psi \text{ is the angle from } \rho_I(p) \text{ to } \rho_R(p) = R_{*p}(\rho_I(R^{-1}p)). \quad (21)$$

Note also that if $\Omega \subseteq S^2$, and $R_0 \in SO(3)$, any smooth function $f_{(R_0)} : U_{R_0} \cap \Omega \rightarrow \mathbb{C}$ determines a unique section f of \mathbf{L}^s over $U_{R_0} \cap \Omega$ with $f_{R_0} = f_{(R_0)}$. Indeed, say $R \in SO(3)$. If $p \in U_{R_0} \cap U_R \cap \Omega$, then $f_R(p)$ must be given by

$$f_R(p) = e^{is\psi} f_{R_0}(p). \quad (22)$$

where of course, $f_{R_0} = f_{(R_0)}$, and where

$$\psi \text{ is the angle from } \rho_{R_0}(p) \text{ to } \rho_R(p). \quad (23)$$

Conversely, there is surely a section f of \mathbf{L}^s over $U_{R_0} \cap \Omega$ with these f_R .

Consequently *the space of smooth sections f of \mathbf{L}^s over Ω may be identified with the space*

$$\{f = (f_R)_{R \in SO(3)} : \text{each } f_R \in C^\infty(U_R \cap \Omega), \text{ and (22), (23) hold for all } R_0, R \in SO(3) \text{ and all } p \in U_{R_0} \cap U_R \cap \Omega\}. \quad (24)$$

We shall frequently make this identification, which often greatly simplifies matters conceptually. However, as is often the case in mathematics, certain properties are clearer, and more easily verified, if one uses the coordinate-free line bundle point of view.

Equation (20) describes what happens when one “rotates the charts”, and we now argue that “you get the same answer if you rotate the section instead”. Thus, if $R \in SO(3)$, as we have seen, $R : S^2 \rightarrow S^2$ gives rise to a smooth map $R_s : \mathbf{L}^s \rightarrow \mathbf{L}^s$. If f is a section of \mathbf{L}^s over $\Omega \subseteq S^2$, using (17) we obtain the “rotated section”

$$f^R = R_s^{-1} \circ f \circ R. \quad (25)$$

over $R^{-1}\Omega$. (Note that if $s = 0$, $f^R = f \circ R$.) Thus, for $p \in \Omega$,

$$(f^R)(R^{-1}p) = R_s^{-1} \circ f(p),$$

from which we see that

$$(f^R)_I(R^{-1}p) = e^{is\psi} f_I(p) \quad (26)$$

where

$$\psi \text{ is the angle from } R_{*p}^{-1}(\rho_I(p)) \text{ to } \rho_I(R^{-1}p). \quad (27)$$

Note that this ψ is the same as the ψ in (21), since R_{*p} preserves angles. Therefore

$$f_R(p) = (f^R)_I(R^{-1}p). \quad (28)$$

One could take the point of view of (24) here, and simply define f^R by setting $(f^R)_I(q) = f_R(Rq)$. But then it would be tedious (though certainly possible) to verify that the smoothness of f implies the smoothness of f^R , or that $(f^{R_1})^{R_2} = f^{R_1 R_2}$, both of which are evident from (25).

We also need to discuss the metric and the orientation we will use on S^2 . Since rotations are holomorphic maps on S^2 , any metric in the intrinsic conformal class on S^2 is rotationally invariant, up to a constant multiple at each point. Thus there is a truly rotationally invariant metric in this conformal class, obtained by rotation of the metric at any particular point on the sphere. We note that such a rotationally invariant metric coincides (up to a constant multiple) with the Euclidean metric, as restricted to tangent vectors on the sphere. (That is, for some positive constant c , if $v = \sum_{j=1}^3 a_j \frac{\partial}{\partial x_j}$ and $w = \sum_{j=1}^3 b_j \frac{\partial}{\partial x_j}$ are real tangent vectors at any point on the sphere, their inner product there is $c < v, w >_E$, where $< v, w >_E = \sum_{j=1}^3 a_j b_j$.) Indeed, since $< v, w >_E$ is rotationally invariant, this fact need only be checked at the South pole \mathbf{S} . We can determine a metric in the intrinsic conformal class on S^2 , by pulling back the standard metric on \mathbb{C} through stereographic projection from the North Pole onto the tangent plane to the sphere at \mathbf{S} ; denote this stereographic projection by T . It is geometrically evident, and it is easily calculated (see the formula (31) below for $\sigma = T/2$), that $T_{*\mathbf{S}}(\partial/\partial x) = \partial/\partial x$, and $T_{*\mathbf{S}}(\partial/\partial y) = \partial/\partial y$. Thus the tangent vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, are, up to a choice of conformal factor at \mathbf{S} , an orthonormal basis at \mathbf{S} , as desired. We use $< v, w >_E$ as our metric on S^2 . We choose the orientation on S^2 which makes T orientation-preserving. Thus $(\partial/\partial x, \partial/\partial y)$ is an oriented orthonormal basis at \mathbf{S} , and, consequently, $(\partial/\partial y, \partial/\partial x)$ is an oriented orthonormal basis at \mathbf{N} .

In our first new result about the spin s bundle over S^2 , we take a section f of \mathbf{L}^s , and examine the behavior of f_I as we approach the north and south poles.

Theorem 3.1 *Let f be a continuous section of \mathbf{L}^s over S^2 . Then the function $e^{is\phi} f_I(\theta, \phi)$ extends continuously from U_I to $U_I \cup \mathbf{N}$, and the function $e^{-is\phi} f_I(\theta, \phi)$ extends continuously from U_I to $U_I \cup \mathbf{S}$.*

Proof Let us show the second statement; the proof of the first statement is almost identical. For any $\phi \in [0, 2\pi]$, let $f_{I,\phi}(\theta) = f_I(\theta, \phi)$. It will suffice to show that $\lim_{\theta \rightarrow \pi^-} e^{-is\phi} f_{I,\phi}(\theta)$ exists, uniformly in ϕ , and that this limit is independent of ϕ .

For $\phi \in [0, 2\pi]$, let γ_ϕ be the great arc $\gamma_\phi(\theta) = (\theta, \phi)$ ($0 \leq \theta \leq \pi$). If $p = \gamma_\phi(\theta)$ is on this great arc, then $\rho_I(p)$ points in the direction $\gamma'_\phi(\theta) \times p$ (cross product); thus

$$\lim_{\theta \rightarrow \pi^-} \rho_I(\gamma_\phi(\theta)) := v_\phi$$

exists, and this limit is uniform in ϕ (indeed, a rotation about \mathbf{N} takes any one of these limiting situations into any other). Moreover, evidently, v_ϕ is a tangent vector at \mathbf{S} , and given our choice of orientation at \mathbf{S} ,

$$\text{the angle from } v_0 \text{ to } v_\phi \text{ is } \phi. \quad (29)$$

Now, let R be any rotation of S^2 which takes \mathbf{S} to a point other than \mathbf{N} or \mathbf{S} . We then have that, for $(\theta, \phi) \in U_I \cap U_R$,

$$f_{I,\phi}(\theta) = e^{-is\psi_{\theta,\phi}} f_R(\theta, \phi)$$

where $\psi_{\theta,\phi}$ is the angle from $\rho_I(p)$ to $\rho_R(p)$, if $p = (\theta, \phi)$. For fixed ϕ , the limit as $\theta \rightarrow \pi^-$ of the right side exists (uniformly in ϕ) and equals $e^{-is\Psi_\phi} f_R(\mathbf{S})$, where now

$$\Psi_\phi \text{ is the angle from } v_\phi \text{ to } \rho_R(\mathbf{S}). \quad (30)$$

Thus for any ϕ ,

$$\lim_{\theta \rightarrow \pi^-} e^{-is\phi} f_{I,\phi}(\theta) = e^{-is(\phi + \Psi_\phi)} f_R(\mathbf{S}) = e^{-is(\phi + \Psi_\phi - \Psi_0)} e^{-is\Psi_0} f_R(\mathbf{S}) = e^{-is(\phi + \Psi_\phi - \Psi_0)} \lim_{\theta \rightarrow \pi^-} f_{I,0}(\theta) = \lim_{\theta \rightarrow \pi^-} f_{I,0}(\theta),$$

since $\Psi_\phi - \Psi_0$ equals $-\phi$ (mod 2π), by (29) and (30). This proves the second statement; the proof of the first statement is entirely similar, if one takes into account the difference in orientation at \mathbf{N} .

4 The Newman-Penrose Theory

In this section we review the spin s theory of Newman and Penrose [39]. We do so within the rigorous framework we have presented in section 3. Most of our arguments are very close to theirs. We resume presenting new results in the next section.

Although we earlier used stereographic projection T from the north pole onto the tangent plane at \mathbf{S} , in what follows we will follow Newman and Penrose and instead use stereographic projection σ from the north pole onto the equatorial plane, from $U_I \subseteq S^2$ to $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Note that $\sigma = T/2$ is still orientation-preserving. Explicitly, if $p \in U_I$ has Euclidean coordinates (x, y, z) , then

$$\sigma p = \frac{1}{1-z}(x + iy), \quad (31)$$

which one easily visualizes by using similar triangles. From this, and the formulas $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$, $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$, it follows that if $p \in U_I$ has standard spherical coordinates (θ, ϕ) , then

$$\sigma p = e^{i\phi} \cot\left(\frac{\theta}{2}\right). \quad (32)$$

The map σ , being a conformal map, preserves angles; we will also need to know the factor by which it multiplies lengths. Thus, if $p \in U_I$, since σ is conformal, there is a number $\lambda_p > 0$ such that for any tangent vector v at p , $|\sigma_{*p}v| = \lambda_p < v, v >_E^{1/2}$. We need to find the conformal factor λ_p . For this, we may assume that $v = \rho_I(p)$ is a unit tangent vector tangent to the circle $\phi = \text{constant}$ through p . Restricted to that circle, by (31), the map σ is simply a dilation by the factor $(1-z)^{-1}$, so we must have $\lambda_p = (1-z)^{-1}$. It is easy to calculate, then, from (31), that $\lambda_p = \frac{1}{2}(1 + |\sigma p|^2)$. For this reason, for $\zeta \in \mathbb{C}$, we set

$$P(\zeta) = \frac{1}{2}(1 + |\zeta|^2), \quad (33)$$

and we have shown that:

$$\text{if } p \in U_I, \sigma p = \zeta, \text{ and } v \text{ is a tangent vector at } p, \text{ then } |\sigma_{*p}v| = P(\zeta) < v, v >_E^{1/2}. \quad (34)$$

We also remark the following fact (which is also geometrically evident). Since σ restricted to the aforementioned circle is just a dilation, if

$$\text{If } p = (\theta, \phi) \in U_I, \text{ then the angle from } \sigma_{*p}\rho_I(p) \text{ to } \partial/\partial y \text{ is } -\phi. \quad (35)$$

Indeed, it equals the angle from $\rho_I(p)$ to $\partial/\partial y$ in \mathbb{R}^3 , which is surely $-\phi$.

We are now ready for the observations of Newman and Penrose, expressed in our language. These observations will be preceded by bullet points in what follows. We will provide rigorous proofs of their observations, within the context of spin s line bundles. Our proofs are modified versions of their arguments.

Say $\Omega \subseteq U_I$ is open. Suppose $h_{(I)} : \Omega \rightarrow \mathbb{C}$ is smooth. Following Newman and Penrose we define

$$\partial_{sI} h_{(I)} = -(\sin \theta)^s \left(\frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right) (\sin \theta)^{-s} h_{(I)}.$$

(Here we are again using standard spherical coordinates (θ, ϕ) on U_I .) Thus $\partial_{sI} h_{(I)}$ is a smooth function on Ω .

Next, if h is a smooth section of \mathbf{L}_Ω^s , we define ∂h to be the section of \mathbf{L}^{s+1} over Ω such that

$$(\partial h)_I = \partial_{sI} h_I.$$

Also, if $\Omega_0 \subseteq \mathbb{C}^*$ is open, and u is a smooth section of $\mathbf{L}_{\Omega_0}^s$, we define the smooth section $\mathcal{D}_s u$ of $\mathbf{L}_{\Omega_0}^{s+1}$ by

$$\mathcal{D}_s u = 2P^{1-s} \frac{\partial P^s u}{\partial \zeta}.$$

(Here P is as in (33). Recall that we have only one chart on \mathbb{C}^* , so sections of $\mathbf{L}_{\Omega_0}^s$ or $\mathbf{L}_{\Omega_0}^{s+1}$ may be naturally identified with smooth functions from Ω_0 to \mathbb{C} .)

If f is a section of \mathbf{L}_{Ω}^s , where $\Omega \subseteq U_I$, let us write $f^{\sigma^{-1}} = f \circ_s \sigma^{-1}$. The first key observation of Newman and Penrose is:

- Suppose f is a smooth section of \mathbf{L}_{Ω}^s , where $\Omega \subseteq U_I$ is open. Then

$$(\partial f)^{\sigma^{-1}} = \mathcal{D}_s(f^{\sigma^{-1}}). \quad (36)$$

To see this, one notes that $\partial_{sI} f_I = s \cot \theta f_I - L f_I$, where $L = (\partial/\partial\theta) + (i/\sin \theta)(\partial/\partial\phi)$. Using (35), we see that the left side of (36) is $e^{-i(s+1)\phi}[(s \cot \theta)f_I \circ \sigma^{-1} - (L f_I) \circ \sigma^{-1}]$. Let $u = f_I \circ \sigma^{-1}$. A brief calculation, using (32), the chain rule, and the fact that $\cot(\theta/2)/\sin \theta = \frac{1}{2} \csc^2(\theta/2)$, shows that $(L f_I) \circ \sigma^{-1} = -e^{i\phi} \csc^2(\theta/2) \partial u / \partial \zeta$. On the other hand, if $p = (\theta, \phi) \in U_I$ and $\zeta = \sigma p$, then $P(\zeta) = \frac{1}{2}[1 + \cot^2(\theta/2)] = \frac{1}{2} \csc^2(\theta/2)$. Thus the left side of (36) equals

$$e^{-i(s+1)\phi} \left[(s \cot \theta)u + 2P e^{i\phi} \frac{\partial u}{\partial \zeta} \right]. \quad (37)$$

On the other hand, the right side of (36) is

$$2P^{1-s} \frac{\partial e^{-is\phi} P^s u}{\partial \zeta}, \quad (38)$$

which equals $2P^{1-s}[\partial e^{-is\phi} P^s / \partial \zeta]u + 2Pe^{-is\phi} \partial u / \partial \zeta$. We are left with showing that

$$2P^{1-s} \frac{\partial e^{-is\phi} P^s}{\partial \zeta} = e^{-i(s+1)\phi} s \cot \theta. \quad (39)$$

Surely $\partial P / \partial \zeta = \frac{1}{2} \bar{\zeta}$. On the other hand, since $e^{2i\phi} = \zeta / \bar{\zeta}$, we find $2ie^{2i\phi} \partial \phi / \partial \zeta = 1 / \bar{\zeta}$, so that $\partial \phi / \partial \zeta = 1 / (2i\zeta)$. Thus the left side of (39) equals

$$e^{-is\phi}[-sP/\zeta + s\bar{\zeta}] = se^{-is\phi}[(|\zeta|^2 - P)/\zeta] = se^{-i(s+1)\phi}[(|\zeta|^2 - P)/\cot(\theta/2)],$$

so all that is left to show is that $(|\zeta|^2 - P)/\cot(\theta/2) = \cot \theta$. But $(|\zeta|^2 - P)/\cot(\theta/2) = [\cot^2(\theta/2) - 1]/[2 \cot(\theta/2)] = \cot \theta$ as desired.

Next, Newman and Penrose observe:

- Say $R \in SO(3)$. Suppose f is a smooth section of \mathbf{L}_{Ω}^s , where $\Omega \subseteq U_I \cap U_R$ is open. Say $R \in SO(3)$. Then

$$\partial(f^R) = (\partial f)^R. \quad (40)$$

(Note that these sections are defined over $R^{-1}\Omega \subseteq U_I$.)

To see (40), it suffices to show that $[\partial(f^R)]^{\sigma^{-1}} = [(\partial f)^R]^{\sigma^{-1}}$. (Note that these sections are defined over $\Omega_1 := \sigma(R^{-1}\Omega)$.) Set $u = f^{\sigma^{-1}}$, defined over $\Omega_0 := \sigma\Omega$, and let

$$\mathcal{I} = \sigma \circ R \circ \sigma^{-1} : \Omega_1 \rightarrow \Omega_0. \quad (41)$$

Then by (36),

$$[\partial(f^R)]^{\sigma^{-1}} = \mathcal{D}_s(u^{\sigma \circ R \circ \sigma^{-1}}) = \mathcal{D}_s u^{\mathcal{I}}.$$

On the other hand,

$$[(\partial f)^R]^{\sigma^{-1}} = (\partial f)^{\sigma^{-1} \circ \sigma \circ R \circ \sigma^{-1}} = (\mathcal{D}_s u)^\mathcal{I}.$$

Thus, we need only show that if u is a section of $L_{\Omega_0}^s \subseteq \mathbb{C}^*$, then

$$\mathcal{D}_s u^\mathcal{I} = (\mathcal{D}_s u)^\mathcal{I}. \quad (42)$$

(Note that these sections are defined over Ω_1 .)

Now \mathcal{I} is a conformal mapping. At any point ζ of Ω_1 , if $\mathcal{I}'(\zeta) = re^{i\psi}$ ($\psi \in [-\pi, \pi]$), then we know that

$$\text{the angle from } \mathcal{I}_{*\zeta}(\partial/\partial y) \text{ to } \partial/\partial y \text{ is } -\psi, \quad (43)$$

while for any real nonzero tangent vector v at ζ , $|\mathcal{I}_{*\zeta}v| = r|v|$. By (34) and (41), we conclude that

$$\mathcal{I}'(\zeta) = \frac{P(\mathcal{I}(\zeta))}{P(\zeta)} e^{i\psi}. \quad (44)$$

Accordingly

$$P(\zeta) e^{i\psi} = \frac{P(\mathcal{I}(\zeta))}{\overline{\mathcal{I}'(\zeta)}}. \quad (45)$$

Turning now to (42), we see by (45) that

$$(\mathcal{D}_s u^\mathcal{I})(\zeta) = 2P(\zeta)^{1-s} \frac{\partial [(P(\zeta)e^{i\psi})^s (u \circ \mathcal{I})(\zeta)]}{\partial \zeta} = \frac{2P(\zeta)}{P(\zeta)^s} \frac{\partial \left[([P^s u] \circ \mathcal{I})(\zeta) / \overline{\mathcal{I}'(\zeta)^s} \right]}{\partial \zeta}.$$

The key point is that $\overline{\mathcal{I}'(\zeta)^s}$ is *antiholomorphic*, so that

$$(\mathcal{D}_s u^\mathcal{I})(\zeta) = \frac{2P(\zeta)}{[P(\zeta)\overline{\mathcal{I}'(\zeta)}]^s} \frac{\partial ([P^s u] \circ \mathcal{I})(\zeta)}{\partial \zeta} = \frac{2P(\zeta)}{P(\mathcal{I}(\zeta))^s e^{-is\psi}} \left[\frac{\partial (P^s u)}{\partial \zeta} \circ \mathcal{I}(\zeta) \right] \mathcal{I}'(\zeta).$$

Finally, then, by (44),

$$(\mathcal{D}_s u^\mathcal{I})(\zeta) = 2P(\mathcal{I}(\zeta))^{1-s} e^{i(s+1)\psi} \left[\frac{\partial (P^s u)}{\partial \zeta} \circ \mathcal{I}(\zeta) \right] = (\mathcal{D}_s u)^\mathcal{I}(\zeta).$$

This proves (42), as desired.

Next, say $\Omega \subseteq U_R$ is open, so that $R^{-1}\Omega \subseteq U_I$. Say $h_{(R)} : \Omega \rightarrow \mathbb{C}$ is smooth; then $h_{(R)} \circ R : R^{-1}\Omega \rightarrow \mathbb{C}$ is smooth, so we can apply ∂_{sI} to it and obtain another smooth function on $R^{-1}\Omega$. Now we define

$$\partial_{sR} h_{(R)} = [\partial_{sI}(h_{(R)} \circ R)] \circ R^{-1}, \quad (46)$$

which is, like $h_{(R)}$ itself, a smooth function on Ω . We now deduce:

- Suppose f is a smooth section of \mathbf{L}^s over an open set $\Omega \subseteq U_I \cap U_R$. Then

$$(\partial f)_R = \partial_{sR} f_R. \quad (47)$$

To see this, say $p \in \Omega$. Then by (28) and (40),

$$(\partial f)_R(p) = [(\partial f)^R]_I(R^{-1}p) = (\partial f^R)_I(R^{-1}p).$$

Thus, by the definition of ∂_{sI} and (28) again,

$$(\partial f)_R(p) = (\partial_{sI} f_I^R)(R^{-1}p) = (\partial_{sI}[f_R \circ R])(R^{-1}p) = (\partial_{sR} f_R)(p),$$

as desired.

We can reformulate the last result more appealingly as follows. On the chart U_R we again use coordinates (θ_R, ϕ_R) , where if the (θ, ϕ) coordinates of $p \in U_I$ are (θ_0, ϕ_0) , then the (θ_R, ϕ_R) coordinates of $Rp \in U_R$ are also (θ_0, ϕ_0) . Then, in these coordinates on U_R ,

$$\partial_{sR} f_R = -(\sin \theta_R)^s \left(\frac{\partial}{\partial \theta_R} + \frac{i}{\sin \theta_R} \frac{\partial}{\partial \phi_R} \right) (\sin \theta_R)^{-s} f_R. \quad (48)$$

So far we have only defined ∂ over the chart U_I , but (47) enables us to define ∂f for any smooth section f of \mathbf{L}_Ω^s , where now Ω is any open subset of S^2 . We simply let ∂f be the smooth section of \mathbf{L}_Ω^{s+1} , with the property that if $p \in \Omega$ is in U_R , then

$$(\partial f)_R(p) = \partial_{sR} f_R(p). \quad (49)$$

Note that this definition makes sense. Indeed, over $\Omega \cap U_I$, this section agrees with the section ∂f we have previously defined. If, instead, p is the north or south pole, choose a particular $R \in SO(3)$ with $p \in U_R$, and let h be the section of \mathbf{L}^s over $\Omega \cap U_R$ such that $h_R = \partial_{sR} f_R$. Then on $\Omega \cap U_R \cap U_I$, h agrees with ∂f as previously defined. Thus, for any other rotation R' such that $p \in U_{R'}$, $h_{R'}(q) = \partial_{sR'} f_{R'}(q)$ for all $q \in \Omega \cap U_R \cap U_{R'} \cap U_I$. Letting $q \rightarrow p$, we see that $h_{R'}(p) = \partial_{sR'} f_{R'}(p)$ as well. This shows that the definition (49) makes sense.

As a consequence we have that:

$$\partial \text{ maps smooth sections of } \mathbf{L}^s \text{ to smooth sections of } \mathbf{L}^{s+1}. \quad (50)$$

Here the sections could be defined over the entire sphere; ∂ is consequently called the “spin-raising operator”.

Note also that, in (40), we may now relax the hypothesis that $\Omega \subseteq U_I \cap U_R$, and only assume that $\Omega \subseteq S^2$ is open. Indeed, both sides of (40) agree on $\Omega \cap U_I \cap U_R$; since both sides are smooth sections of \mathbf{L}_Ω^s , they must agree on all of Ω .

As is customary for sections of line bundles, we let $C^\infty(\mathbf{L}^s)$ denote the space of smooth sections of \mathbf{L}^s , defined over all of S^2 . (In section 2, we called this space $C_s^\infty(S^2)$.) Newman and Penrose make the following important observation:

- If $s < 0$, then $\partial : C^\infty(\mathbf{L}^s) \rightarrow C^\infty(\mathbf{L}^{s+1})$ is injective. (51)

To see this, say $f \in C^\infty(\mathbf{L}^s)$, and that $\partial f = 0$. By Theorem 3.1, $|f_I|$ is bounded on U_I . Let $u = f_I \circ \sigma^{-1}$; then u is bounded on \mathbb{C}^* . As in (36) and (38), we see that

$$0 = (\partial f)^{\sigma^{-1}} = \mathcal{D}_s(f^{\sigma^{-1}}) = 2P^{1-s}[\partial e^{-is\phi} P^s u / \partial \zeta].$$

The function $v := e^{is\phi} P^s \bar{u}$ is therefore holomorphic on \mathbb{C}^* . It is bounded near 0, so it extends holomorphically to \mathbb{C} . But, since $s < 0$, $[P(\zeta)]^s \rightarrow 0$ as $\zeta \rightarrow \infty$, $v \equiv 0$ by Liouville’s theorem. Consequently u , and hence f , are both zero, as desired.

Say now that $\Omega \subseteq S^2$ is open. There is evidently a well-defined map of *complex conjugation* from $C^\infty(\mathbf{L}_\Omega^s)$ to $C^\infty(\mathbf{L}_\Omega^{-s})$, taking a section f in the former space to a section \bar{f} in the latter space, defined by

$$(\bar{f})_R(p) = \overline{f_R(p)}$$

for all $p \in U_R \cap \Omega$ (for any $R \in SO(3)$). We may then define the “spin-lowering operator” $\bar{\partial}$ by

$$\bar{\partial} f = \overline{\partial \bar{f}}, \quad (52)$$

so that $\bar{\partial} : C^\infty(\mathbf{L}_\Omega^s) \rightarrow C^\infty(\mathbf{L}_\Omega^{s-1})$. In the coordinates (θ_R, ϕ_R) on U_R , we evidently have that

$$(\bar{\partial} f)_R = \overline{\partial_{sR} f_R},$$

where

$$\bar{\partial}_{sR} f_R = -(\sin \theta_R)^{-s} \left(\frac{\partial}{\partial \theta_R} - \frac{i}{\sin \theta_R} \frac{\partial}{\partial \phi_R} \right) (\sin \theta_R)^s f_R. \quad (53)$$

If $\Omega_0 \subseteq \mathbb{C}^*$ is open, and u is a smooth section of $\mathbf{L}_{\Omega_0}^s$, we define the smooth section $\bar{\mathcal{D}}_s u$ of $\mathbf{L}_{\Omega_0}^{s-1}$ by $\bar{\mathcal{D}}_s u = \bar{\mathcal{D}}_s \bar{u}$; thus

$$\bar{\mathcal{D}}_s u = 2P^{1+s} \frac{\partial P^{-s} u}{\partial \bar{\zeta}}.$$

By taking complex conjugates in (36), (40), and (51), we find:

- Suppose f is a smooth section of \mathbf{L}_Ω^s , where $\Omega \subseteq U_I$ is open. Then

$$(\bar{\partial} f)^{\sigma^{-1}} = \bar{\mathcal{D}}_s (f^{\sigma^{-1}}). \quad (54)$$

- Suppose f is a smooth section of \mathbf{L}_Ω^s , where $\Omega \subseteq S^2$ is open. Say $R \in SO(3)$. Then

$$\bar{\partial}(f^R) = \bar{\partial} f^R. \quad (55)$$

- If $s > 0$, then $\bar{\partial} : C^\infty(\mathbf{L}^s) \rightarrow C^\infty(\mathbf{L}^{s-1})$ is injective. (56)

One can define L^2 sections of \mathbf{L}^s , (or on any other line bundle over a compact oriented manifold, for that matter), and define an L^2 norm on them, by using a finite atlas on the bundle and a partition of unity on the manifold. Different choices of atlas and partition of unity lead to the same space of L^2 sections with an equivalent L^2 norm. For the bundle \mathbf{L}^s there is a very natural choice of L^2 norm, equivalent to the L^2 norm obtained through such a construction. Note first that if f is an L^2 section of \mathbf{L}^s , then (a representative of) f is defined almost everywhere. Say f is defined at $p \in U_R \cap U_{R'}$. We may then write $\Phi_R(f(p)) = (p, f_R(p))$, $\Phi_{R'}(f(p)) = (p, f_{R'}(p))$, $f_{R'}(p) = e^{is\psi} f_R(p)$, where ψ is the angle from $\rho_R(p)$ to $\rho_{R'}(p)$. Thus $|f_R(p)|^2$ is independent of the choice of R , as long as $p \in U_R$. Accordingly

$$|f(p)|^2 := |f_R(p)|^2 \quad (p \in U_R)$$

is well-defined a.e. on S^2 . We then define the L^2 norm of f by

$$\|f\|_{L^2}^2 = \int_{S^2} |f|^2$$

where the integral is with respect to the usual surface measure on S^2 . We let $L^2(\mathbf{L}^s)$ denote the space of L^2 sections of \mathbf{L}^s , with this norm.

We may now show:

- The formal adjoint of $\partial : C^\infty(\mathbf{L}^s) \rightarrow C^\infty(\mathbf{L}^{s+1})$ is $-\bar{\partial}$.

For this, we need to show that, whenever $f \in C^\infty(\mathbf{L}^s)$ and $g \in C^\infty(\mathbf{L}^{s+1})$, we have

$$\langle \partial f, g \rangle = - \langle f, \bar{\partial} g \rangle, \quad (57)$$

where $\langle \cdot, \cdot \rangle$ denotes L^2 inner product.

For this, we first note the following useful proposition:

Proposition 4.1 (a) Let N be a positive integer. Then S^2 is not the union of N open subsets of diameter less than π/N .

(b) Any open set $\Omega \subseteq S^2$ of diameter less than $\pi/2$ is contained in some U_R .

Proof For (a), say it were, and let U_1 be one of those open subsets. By connectedness of S^2 , U_1 must intersect another one of those sets, say U_2 . Again by connectedness, $U_1 \cup U_2$ must intersect another one of those sets, say U_3 . Continuing in this manner, we obtain the contradiction

$$\pi = \text{diam} S^2 \leq \text{diam} U_1 + \text{diam} U_2 + \dots < N\pi/N.$$

For (b), we need to show that Ω omits some pair of antipodal points in S^2 . If it did not, then if A is the antipodal map, $\Omega \cup A\Omega = S^2$. This contradicts (a).

Let us then cover S^2 by a finite collection of open sets of diameter less than $\pi/5$, choose a partition of unity $\{\zeta_j\}$ subordinate to this cover, and break up $f = \sum \zeta_j f$ and $g = \sum \zeta_j g$. By Proposition 4.1 (b), by using this breakup, in proving (57), we may assume that $\text{supp } f \cup \text{supp } g \subseteq U_R$ for some R . Then we need to show that

$$\int_0^{2\pi} \int_{-\pi}^{\pi} [(\partial_{sR} f_R) \overline{g_R}] (\theta_R, \phi_R) \sin \theta_R d\theta_R d\phi_R = - \int_0^{2\pi} \int_{-\pi}^{\pi} [f_R \overline{\partial_{s+1,R} g_R}] (\theta_R, \phi_R) \sin \theta_R d\theta_R d\phi_R.$$

But this is an elementary calculation, using (48) and (53).

We are now ready (again following Newman and Penrose) to define the spin s spherical harmonics. We let $\{Y_{lm} : 0 \leq l, -l \leq m \leq l\}$ be the usual orthonormal basis of spherical harmonics for S^2 ,

$$Y_{lm}(\theta, \phi) = e^{im\phi} (-1)^{m^-} \frac{a_{lm}}{l!} \sin^{2l}(\theta/2) \sum_r \binom{l}{r} \binom{l}{r-m} (-1)^{l-r} \cot^{2r-m}(\theta/2). \quad (58)$$

where

$$a_{lm} = [(l+m)!(l-m)!(2l+1)/(4\pi)]^{\frac{1}{2}},$$

and our convention is that the binomial coefficient $\binom{n}{j}$ is zero if $j < 0$ (or $n < 0$). (As usual $m^+ = \max(m, 0)$, $m^- = \max(-m, 0)$. The factor of $(-1)^{m^-}$ in (58) is usually not included, but by including it, we have $\overline{Y_{lm}} = Y_{l,-m}$, which will simplify many formulas in the sequel.)

For $l \geq |s|$, Newman and Penrose then define the spin s spherical harmonic ${}_s Y_{lm}$ by

$${}_s Y_{lm} = \begin{cases} \left[\frac{(l-s)!}{(l+s)!} \right]^{\frac{1}{2}} \partial^s Y_{lm} & (0 \leq s \leq l), \\ \left[\frac{(l+s)!}{(l-s)!} \right]^{\frac{1}{2}} (-\bar{\partial})^{-s} Y_{lm} & (-l \leq s \leq 0). \end{cases} \quad (59)$$

The ${}_s Y_{lm}$ are not defined for $l < |s|$. Note that ${}_s Y_{lm}$ is a smooth section of \mathbf{L}^s . We have ([39], [22]):

$${}_s Y_{lmI}(\theta, \phi) = e^{im\phi} (-1)^{m^-} \frac{a_{lm}}{[(l+s)!(l-s)!]^{1/2}} \sin^{2l}(\theta/2) \sum_r \binom{l-s}{r} \binom{l+s}{r+s-m} (-1)^{l-r-s} \cot^{2r+s-m}(\theta/2). \quad (60)$$

To see this, let ${}_s v_{lm}$ denote the section of \mathbf{L}^s over U_I such that ${}_s v_{lmI}$ equals the right side of (60). We need to show that ${}_s v_{lm} = {}_s Y_{lm}$ over U_I . Note that this is surely true when $s = 0$. Let ${}_s u_{lm} = {}_s v_{lm}^{\sigma^{-1}}$. By (36), when $s \geq 0$, we need only show that

$$\mathcal{D}^s {}_0 u_{lm} = \left[\frac{(l+s)!}{(l-s)!} \right]^{\frac{1}{2}} {}_s u_{lm}. \quad (61)$$

(Here $\mathcal{D}^s = \mathcal{D}_{s-1} \circ \dots \circ \mathcal{D}_0$.) However, using (35) again, one easily calculates that, whenever $l \geq |s|$,

$${}_s u_{lm}(\zeta) = \frac{(-1)^{l-m^+} a_{lm}}{[(l+s)!(l-s)!]^{1/2}} (1 + |\zeta|^2)^{-l} \sum_r \binom{l-s}{r} \binom{l+s}{r+s-m} \zeta^r (-\bar{\zeta})^{r+s-m}. \quad (62)$$

From this, a brief calculation using Pascal's rule and the identities

$$(l-s-r) \binom{l-s}{r} = (r+1) \binom{l-s}{r+1} = (l-s) \binom{l-s-1}{r}$$

shows that, whenever $l \geq |s|$,

$$\mathcal{D}_s {}_s u_{lm}(\zeta) = [(l-s)(l+s+1)]^{\frac{1}{2}} {}_{s+1} u_{lm}. \quad (63)$$

(Here, our convention is, that since the factor $(l-s)$ appears on the right side of (63), the right side is to be interpreted as zero if $l=s$.) From this, (61) follows at once by induction on s , for $s \geq 0$, and so we have (60) for $s \geq 0$ as well. We then obtain (60) for $s < 0$ by complex conjugation and use of the identity

$$\overline{{}_s Y_{lm}} = {}_{-s} Y_{l,-m,I}.$$

Even better, by (63) and (36), we have that, whenever $l \geq |s|$,

$$\partial_s {}_s Y_{lm}(\zeta) = [(l-s)(l+s+1)]^{\frac{1}{2}} {}_{s+1} Y_{lm}. \quad (64)$$

By taking complex conjugates of this, we also find that, whenever $l \geq |s|$,

$$\overline{\partial}_s {}_s Y_{lm}(\zeta) = -[(l+s)(l-s+1)]^{\frac{1}{2}} {}_{s-1} Y_{lm}. \quad (65)$$

Newman and Penrose next observe that, for each s , the $\{{}_s Y_{lm}\}$ are orthonormal. Indeed, set

$$b_{ls} = \begin{cases} [(l+s)!/(l-s)!]^{1/2} & \text{if } 0 \leq s \leq l, \\ [(l-s)!/(l+s)!]^{1/2} & \text{if } -l \leq s \leq 0. \end{cases} \quad (66)$$

By (64), if $s > 0$,

$${}_s Y_{lm} = \frac{1}{b_{ls}} \partial^s Y_{lm}, \quad Y_{lm} = \frac{1}{b_{ls}} (-\overline{\partial})^s {}_s Y_{lm}, \quad (67)$$

while if $s < 0$,

$${}_s Y_{lm} = \frac{1}{b_{ls}} (-\overline{\partial})^{-s} Y_{lm}, \quad Y_{lm} = \frac{1}{b_{ls}} \partial^{-s} {}_s Y_{lm}. \quad (68)$$

If $s \geq 0$, we see that

$$\langle {}_s Y_{lm}, {}_s Y_{l'm'} \rangle = \frac{1}{b_{ls}} \langle \partial^s Y_{lm}, {}_s Y_{l'm'} \rangle = \frac{1}{b_{ls}} \langle Y_{lm}, (-\overline{\partial})^s {}_s Y_{l'm'} \rangle = \langle Y_{lm}, Y_{l'm'} \rangle = \delta_{ll'} \delta_{mm'},$$

which proves the orthonormality. If $s < 0$, one obtains the orthonormality through complex conjugation. Thus, the $\{{}_s Y_{lm}\}$ are an orthonormal set of eigenfunctions for the (formally) positive self-adjoint operator $-\overline{\partial}\partial : C^\infty(\mathbf{L}^s) \rightarrow C^\infty(\mathbf{L}^s)$. Since this operator is elliptic (by (48) and (53)), it has an orthonormal basis of smooth eigenfunctions, so at this point one would surely expect:

- The $\{{}_s Y_{lm}\}$ are an orthonormal basis of smooth eigenfunctions for $-\overline{\partial}\partial : C^\infty(\mathbf{L}^s) \rightarrow C^\infty(\mathbf{L}^s)$. (69)

This would follow if we knew:

- If $s > 0$, then $\partial^s : C^\infty \rightarrow C^\infty(\mathbf{L}^s)$ is surjective. (70)

Indeed, say $s > 0$. Now $\partial^s : C^\infty \rightarrow C^\infty(\mathbf{L}^s)$ is continuous (using their Fréchet space topologies). (As usual, this topology is obtained by using a finite atlas on the bundle and a partition of unity on the manifold. Different choices of atlas and partition of unity lead to an equivalent Fréchet space topology.) Thus, since the linear span of the $\{Y_{lm}\}$ is dense in C^∞ , the linear span of the $\{{}_s Y_{lm}\}$ is dense in $\partial^s C^\infty = C^\infty(\mathbf{L}^s)$, which is in turn L^2 -dense in the space of L^2 sections. This would prove (69) for $s > 0$ (and hence for all s by complex conjugation), if we knew (70). Moreover, (70) itself follows from the injectivity of $\overline{\partial}^s : C^\infty(\mathbf{L}^s) \rightarrow C^\infty$ (see (56)), and by use of elliptic theory and Sobolev spaces.

Rather than proceed in this manner, and in order to extract important additional information, we give a direct proof of (69) and (70), by following the method of Newman and Penrose. Their method was somewhat formal, but it can easily be made rigorous. In fact, we shall show:

Theorem 4.2 Fix an integer s .

(a) Let $\{\|\cdot\|_N\}$ be a family of norms defining the Fréchet topology on $C^\infty(\mathbf{L}^s)$. Then for any N there exist C, M such that

$$\|{}_s Y_{lm}\|_N \leq C(l+1)^M \quad (71)$$

whenever $l \geq |s|$.

(b) For every $f \in C^\infty(\mathbf{L}^s)$, there exist numbers $\{A_{lm} : l \geq |s|, -l \leq m \leq l\}$ such that

$$f = \sum_{l \geq |s|} \sum_m A_{lm} {}_s Y_{lm} \quad (72)$$

(convergence in $C^\infty(\mathbf{L}^s)$), such that the A_{lm} decay rapidly, by which we mean that for every N there exist C, M such that

$$|A_{lm}| \leq C(l+1)^{-M} \quad (73)$$

for all l, m . Further, we must have $A_{lm} = \langle f, {}_s Y_{lm} \rangle$.

Proof (a) is well-known when $s = 0$. It then follows for all s by use of (59), as well as (53) and (48) for general R . Note, then, that any series as on the right side of (72), where the A_{lm} decay rapidly, does converge in $C^\infty(\mathbf{L}^s)$.

(b) is also well-known when $s = 0$. To prove it in general, we may as usual assume $s > 0$. In that case we may write

$$\bar{\partial}^s f = \sum_{l \geq 0} \sum_m B_{lm} Y_{lm} = \sum_{l \geq s} \sum_m B_{lm} Y_{lm} + g$$

where the B_{lm} decay rapidly, and where $g = \sum_{l < s} \sum_m B_{lm} Y_{lm} \in \ker \partial^s$. Now, with b_{ls} as in (66), set $A_{lm} = (-1)^s B_{lm}/b_{ls}$ (for $l \geq |s|$). Then, again using (a) for $s = 0$, we find

$$\bar{\partial}^s (f - \sum_{l \geq s} \sum_m A_{lm} {}_s Y_{lm}) = g \in \ker \partial^s. \quad (74)$$

Now it is a simple general fact, that if D is a differential operator mapping smooth sections of one hermitian line bundle, over a compact oriented manifold, to smooth sections of another hermitian line bundle over that manifold, and if DF is in the kernel of D^* for some smooth F , then $DF = 0$. (Indeed, $D^*DF = 0$, so the inner product of DF with itself is zero.) Thus the left side of (74) is zero, and so we have (72), by (56). Surely $A_{lm} = \langle f, {}_s Y_{lm} \rangle$, by the orthonormality of the ${}_s Y_{lm}$. This completes the proof. Moreover, (70) follows as well, since if $s > 0$,

$$f = \partial^s (\sum_{l \geq s} \sum_m C_{lm} Y_{lm}),$$

if $C_{lm} = A_{lm}/b_{ls}$.

As we have said, the completeness of the $\{{}_s Y_{lm}\}$ as an orthonormal basis follows from (70). It also follows directly from Theorem 4.2 (b), which shows that the linear span of the $\{{}_s Y_{lm}\}$ is dense in $C^\infty(\mathbf{L}^s)$.

If $s > 0$, we may write

$$C^\infty = V \bigoplus W \quad (75)$$

where

$$V = \overline{\partial}^s C^\infty(\mathbf{L}^s) = \overline{\langle Y_{lm} : l \geq s \rangle} \quad (76)$$

and

$$W = \ker \partial^s = \langle Y_{lm} : l < s \rangle. \quad (77)$$

Moreover,

$$\partial^s : V \rightarrow C^\infty(\mathbf{L}^s) \text{ bijectively.} \quad (78)$$

The statements (75) and (78) are evident, once one identifies V and W as spaces spanned by certain spherical harmonics, as in (76) and (77). One could obtain (75) and (78) by using only the first definitions of V and W in (76) and (77), by using elliptic theory and (56). However, by identifying them as spaces spanned by certain spherical harmonics, Newman and Penrose observe a crucial additional structure. Namely, the decomposition (75) respects the splitting of a function into its real and imaginary parts. That is, if $g \in V$ (or W), so are $\Re g$ and $\Im g$. (This is evident, since $\overline{Y_{lm}} = Y_{l,-m}$.) This splitting of any function in V then induces, through the isomorphism in (78), a corresponding splitting of any element of $C^\infty(\mathbf{L}^s)$. Thus, say $f \in C^\infty(\mathbf{L}^s)$. Suppose $s > 0$. We write $f = \partial^s g$ for some $g \in C^\infty$, then set

$$f_{\mathbf{E}} = \partial^s \Re g, \quad f_{\mathbf{M}} = \partial^s \Im g.$$

$f_{\mathbf{E}}$ and $f_{\mathbf{M}}$ are well-defined, independent of choice of g . Indeed, if $\partial^s h = 0$, then $h \in W$, so $\Re h$ and $\Im h$ are in W , so $\partial^s \Re h = \partial^s \Im h = 0$. In particular, we could let g be the unique solution of $f = \partial^s g$ in V . Similarly, if $s < 0$, we write $f = \overline{\partial}^s g$ for some $g \in C^\infty$, then set

$$f_{\mathbf{E}} = \overline{\partial}^s \Re g, \quad f_{\mathbf{M}} = \overline{\partial}^s \Im g.$$

Surely we always have $f = f_{\mathbf{E}} + i f_{\mathbf{M}}$.

Again suppose $s > 0$. By repeated use of (55), one sees that the space V is rotationally invariant, and then by repeated use of (40), one sees that

$$(f^R)_{\mathbf{E}} = (f_{\mathbf{E}})^R, \quad (f^R)_{\mathbf{M}} = (f_{\mathbf{M}})^R, \quad (79)$$

for any $R \in SO(3)$. Similarly these relations hold for $s < 0$ as well.

In fact, if f is as in (72), it is very easy to find the expansions of $f_{\mathbf{E}}$ and $f_{\mathbf{M}}$ in terms of the $_s Y_{lm}$. Say first $s > 0$. The solution g of $f = \partial^s g$, with $g \in V$, is simply

$$\sum_{l \geq s} \sum_m \frac{A_{lm}}{b_{ls}} {}_s Y_{lm}.$$

A very brief calculation, using $\overline{Y_{lm}} = Y_{l,-m}$, now shows that

$$f_{\mathbf{E}} = \sum_{l \geq |s|} \sum_m A_{lm} {}_{\mathbf{E}} {}_s Y_{lm} \quad (80)$$

and

$$f_{\mathbf{M}} = \sum_{l \geq |s|} \sum_m A_{lm} {}_{\mathbf{M}} {}_s Y_{lm} \quad (81)$$

where

$$A_{lm} {}_{\mathbf{E}} = (A_{lm} + \overline{A_{l,-m}})/2, \quad (82)$$

while

$$A_{lm} {}_{\mathbf{M}} = -i(A_{lm} - \overline{A_{l,-m}})/2. \quad (83)$$

Similarly, (80), (81), (82) and (83) hold for $s < 0$ as well. Up to normalization constants, these formulas agree with formulas in section II of [53], and also with equation (54) of [7] (to see the latter, use equations (55), (65), (66) and (69) of that article). (Those articles take $s = 2$ and write $f = Q + iU$. [7] uses the notation $a_{(lm)}^G$ and $a_{(lm)}^C$ in place of our $A_{lm} {}_{\mathbf{E}}$ and $A_{lm} {}_{\mathbf{M}}$.)

Let us now let

$$\Delta_s = \begin{cases} -\overline{\partial} \partial & \text{if } s \geq 0, \\ -\partial \overline{\partial} & \text{if } s < 0. \end{cases} \quad (84)$$

(Here Δ_s acts on smooth sections of \mathbf{L}^s .) We also let

$$\lambda_{ls} = \begin{cases} (l-s)(l+s+1) & \text{if } 0 \leq s \leq l, \\ (l+s)(l-s+1) & \text{if } -l \leq s < 0. \end{cases} \quad (85)$$

We also let

$$\mathcal{H}_{ls} = \langle {}_s Y_{lm} : -l \leq m \leq l \rangle \quad (\text{for } l \geq |s|). \quad (86)$$

Also

$$P_{ls} \text{ denotes the projection in } L^2(\mathbf{L}^s) \text{ onto } \mathcal{H}_{ls}. \quad (87)$$

Then $L^2(\mathbf{L}^s) = \bigoplus_{l \geq |s|} \mathcal{H}_{ls}$. We also have that \mathcal{H}_{ls} is the space of smooth sections of \mathbf{L}^s which are eigenfunctions of Δ_s for the eigenvalue λ_{ls} . (This follows from (64), (65), and Theorem 4.2.) (In particular, then, Δ_0 is the usual spherical Laplacian on S^2 .) This was all noted by Newman and Penrose.

Of course $\{{}_s Y_{lm} : -l \leq m \leq l\}$ is an orthonormal basis for \mathcal{H}_{ls} . Note that the map $f \rightarrow f^{R^{-1}} = R_s \circ f \circ R^{-1}$ is a *unitary mapping* of $L^2(\mathbf{L}^s)$ onto itself, for any $R \in SO(3)$. Note also that for each such R , ${}_s Y_{lm}^{R^{-1}}$ is an eigenfunction of Δ_s for the eigenvalue λ_{ls} . (This follows from (40) and (55).) Thus, for any R , $\{{}_s Y_{lm}^{R^{-1}} : -l \leq m \leq l\}$ is also an orthonormal basis of \mathcal{H}_{ls} . Note also that, by (28), we have, for $p \in U_R$,

$$({}_s Y_{lm}^{R^{-1}})_R(p) = {}_s Y_{lmI}(R^{-1}p). \quad (88)$$

and so, in the (θ_R, ϕ_R) coordinates on U_R , we have

$$({}_s Y_{lm}^{R^{-1}})_R(\theta_R, \phi_R) = e^{im\phi_R} (-1)^{m^-} \frac{a_{lm}}{[(l+s)!(l-s)!]^{1/2}} \sin^{2l}(\theta_R/2) \sum_r \binom{l-s}{r} \binom{l+s}{r+s-m} (-1)^{l-r-s} \cot^{2r+s-m}(\theta_R/2). \quad (89)$$

5 Zonal Harmonics and Kernels

We now return to our new results.

Concerning the ${}_s Y_{lm}$, we also make the following observation, which is essentially in [7], section 6.

Proposition 5.1

$$\lim_{\theta \rightarrow 0^+} e^{is\phi} {}_s Y_{lmI}(\theta, \phi) = \begin{cases} (-1)^{s^+} \left[\frac{2l+1}{4\pi} \right]^{1/2} & \text{if } m = -s \\ 0 & \text{otherwise} \end{cases}$$

Proof Since ${}_s Y_{lmI}(\theta, \phi)$ equals $e^{im\phi}$ times a function of θ , this limit (which surely exists, by Theorem 3.1) must be zero if $m \neq -s$. If $m = -s$, it follows from an examination of (58). (In each term in the summation in (60), because of the binomial coefficients, we must have $r \leq l-s$. As $\theta \rightarrow 0^+$, $\sin^{2l}(\theta/2) \cot^{2r+s-m}(\theta/2) = \sin^{2l}(\theta/2) \cot^{2r+2s}(\theta/2)$ has a nonzero limit only if $2l \leq 2r+2s$. These two conditions together allow us to look only at the term in which $r = l-s$ when taking the limit, and the proposition now follows at once.)

We next discuss the *kernel* of the projection operator P_{ls} ; we claim that it is a *smooth-kernel operator*. Since P_{ls} maps sections in $L^2(\mathbf{L}^s)$ to smooth sections of \mathbf{L}^s , we must define what we mean by the kernel of such an operator. For $x \in S^2$, let \mathbf{L}_x^s be $\pi^{-1}(x)$, the fiber above x .

Definition 5.2 *We say that $T : L^2(\mathbf{L}^s) \rightarrow C^\infty(\mathbf{L}^s)$ is a smooth-kernel operator if T has the form*

$$(Tf)(x) = \int_{S^2} K(x, y) f(y) dS(y) \quad (90)$$

for any $f \in L^2(\mathbf{L}^s)$; here we require:

- (i) For each $(x, y) \in S^2 \times S^2$, $K(x, y) : \mathbf{L}_y^s \rightarrow \mathbf{L}_x^s$ is linear;
- (ii) $K(x, y)$ depends smoothly on $(x, y) \in S^2 \times S^2$.

We also call the restriction of T to $C^\infty(\mathbf{L}^s)$ a smooth-kernel operator. We call the collection of linear maps $K = \{K(x, y) : (x, y) \in S^2 \times S^2\}$ the kernel of T .

Note here that the integral in (90) is a vector-valued integral. Also note that the statement (ii) means that the kernel is smooth in (x, y) after one locally trivializes the bundle.

To say that T is a smooth-kernel operator is evidently equivalent to saying the following: for each $R, R' \in SO(3)$, there exists a function $K_{R', R} \in C^\infty(U_{R'} \times U_R)$, such that for all smooth sections f of \mathbf{L}^s with compact support in U_R , we have

$$(Tf)_{R'}(x) = \int_{S^2} K_{R', R}(x, y) f_R(y) dS(y), \quad (91)$$

for all $x \in U_{R'}$.

Evidently, if $R, R', R_1, R'_1 \in SO(3)$, if (91) holds, we must have

$$K_{R_1', R_1}(x, y) = e^{is(\psi_1 - \psi_2)} K_{R', R}(x, y) \quad (92)$$

for all $x \in U_{R_1'} \cap U_{R'}$ and $y \in U_{R_1} \cap U_R$, where

$$\psi_1 \text{ is the angle from } \rho_{R'}(x) \text{ to } \rho_{R_1'}(x), \text{ and } \psi_2 \text{ is the angle from } \rho_R(y) \text{ to } \rho_{R_1}(y). \quad (93)$$

Let us now examine the kernel of the projection operator P_{ls} . Evidently, if $f \in L^2(\mathbf{L}^s)$, we have

$$P_{ls} f = \sum_{m=-l}^l \langle f, {}_s Y_{lm} \rangle {}_s Y_{lm}. \quad (94)$$

(Here one could use ${}_s Y_{lm}^{R_0^{-1}}$ in place of ${}_s Y_{lm}$, for any $R_0 \in SO(3)$.) Thus P_{ls} has kernel $K_{R', R}^{ls}$, where for any $R, R' \in SO(3)$,

$$K_{R', R}^{ls}(x, y) = \sum_{m=-l}^l {}_s Y_{lmR'}(x) \overline{{}_s Y_{lmR}(y)}. \quad (95)$$

(Again here, one could use ${}_s Y_{lm}^{R_0^{-1}}$ in place of ${}_s Y_{lm}$, for any $R_0 \in SO(3)$.)

Definition 5.3 For $|l| \geq s$, we define ${}_s Z_l$, the “ s -zonal harmonic” in \mathcal{H}_{ls} , by

$${}_s Z_l = (-1)^{s+} \left[\frac{2l+1}{4\pi} \right]^{1/2} {}_s Y_{l,-s}. \quad (96)$$

We have the following new theorem, which provides justification for this definition. When $s = 0$ this theorem reduces to the well-known result that

$${}_0 Z_l(x) := Z_l(x) = K^{l0}(x, \mathbf{N}) \text{ is the zonal harmonic in } \mathcal{H}_{l0}. \quad (97)$$

Theorem 5.4 (a) Say $\mathbf{N} \in U_R$, and that $x \in U_I \cap U_{R'}$. Then

$${}_s Z_{II}(x) = e^{is(\psi_1 - \psi_2)} K_{R',R}^{ls}(x, \mathbf{N}), \quad (98)$$

where

$$\psi_1 \text{ is the angle from } \rho_{R'}(x) \text{ to } \rho_I(x), \text{ and } \psi_2 \text{ is the angle from } \rho_R(\mathbf{N}) \text{ to } \partial/\partial y \text{ (at } \mathbf{N}). \quad (99)$$

(b) For any $f \in \mathcal{H}_{ls}$, we have

$$\lim_{\theta \rightarrow 0} e^{is\phi} f_I(\theta, \phi) = \langle f, {}_s Z_l \rangle. \quad (100)$$

Proof For (a), note that if $y \in U_I \cap U_R$, then $K_{I,I}^{ls}(x, y)$ equals

$$\sum_m {}_s Y_{lmI}(x) \overline{{}_s Y_{lmI}(y)} = e^{is(\psi_1 - \psi_2)} K_{R',R}^{ls}(x, y)$$

where ψ_1 is the angle from $\rho_{R'}(x)$ to $\rho_I(x)$, and ψ_2 is the angle from $\rho_R(y)$ to $\rho_I(y)$. In this equation, let us say y has coordinates (θ, ϕ) (in the usual coordinates on U_I). Let us fix ϕ , multiply both sides of the equation by $e^{-is\phi}$, and take the limit as $\theta \rightarrow 0^+$. (One can, if one wishes, choose to take $\phi = 0$; or one can let ϕ be arbitrary, and recall our choice of orientation at \mathbf{N} .) (a) now follows at once from Proposition 5.1.

To prove (b), one may assume $f = {}_s Y_{lm}$ for some m ; but then the result is evident from Proposition 5.1. (Alternatively, one can start with the equation $f_I(x) = \int_{S^2} K_{I,I}^{ls}(x, y) f_I(y) dS(y)$ (which is valid for $f \in \mathcal{H}_{ls}$, $x \in U_I$), multiply by $e^{is\phi}$, and take a limit.) This completes the proof.

For another new result, also familiar when $s = 0$, we examine $K_{R,R}^{ls}(x, x)$, for $x \in U_R$. It is evident from (92), (93) that this quantity is independent of R . In fact it is independent of x as well:

Theorem 5.5 If $x \in U_R$, then

$$K_{R,R}^{ls}(x, x) = \frac{2l+1}{4\pi}. \quad (101)$$

Proof It is enough to prove this for $R = I$, since we will then know it unless $x = \mathbf{N}$ or \mathbf{S} , and the result will follow in those cases too by continuity. Let us then take $R = I$ and free the letter R for other uses. Say now $x, y \in U_I$. Choose $R \in SO(3)$ with $Ry = x$. Then $x \in U_I \cap U_R$, so

$$K_{I,I}^{ls}(x, x) = K_{R,R}^{ls}(x, x) = \sum_m |({}_s Y_{lm}^{R^{-1}})_R(x)|^2 = \sum_m |{}_s Y_{lmI}(R^{-1}x)|^2 = K_{I,I}^{ls}(y, y).$$

(We used (88).) Accordingly $K_{I,I}^{ls}(x, x)$ is independent of $x \in SO(3)$. Thus for any fixed $x \in SO(3)$,

$$4\pi K_{I,I}^{ls}(x, x) = \int_{S^2} K_{I,I}^{ls}(y, y) dS(y) = \sum_m \int_{S^2} |{}_s Y_{lm}(y)|^2 dS(y) = 2l+1,$$

as desired.

In doing estimates involving the K^{ls} , it will be very useful to observe the theorem which follows. In this theorem, $\partial_R^{[s]}$ will denote the differential operator on U_R which satisfies $(\partial^s F)_R = \partial_R^{[s]} F_R$, for any ordinary smooth function F on U_R ; explicitly, $\partial_R^{[s]} = \partial_{s-1,R} \circ \dots \circ \partial_{0,R}$. Similarly, $\bar{\partial}_R^{[s]}$ will denote the differential operator on U_R which satisfies $(\bar{\partial}^s F)_R = \bar{\partial}_R^{[s]} F_R$, for any ordinary smooth function F on U_R .

Theorem 5.6 Say $x \in U_{R'}$ and $y \in U_R$. Then if $s > 0$,

$$K_{R',R}^{ls}(x, y) = \frac{(l-s)!}{(l+s)!} \partial_{R',x}^{[s]} \bar{\partial}_{R,y}^{[s]} K^{l0}(x, y), \quad (102)$$

while if $s < 0$,

$$K_{R',R}^{ls}(x,y) = \frac{(l+s)!}{(l-s)!} \bar{\partial}_{R',x}^{[-s]} \partial_{R,y}^{[-s]} K^{l0}(x,y). \quad (103)$$

(Here $\bar{\partial}_{R',x}^{[s]}$ denotes $\partial_{R'}^{[s]}$ taken in the x variable, etc.)

Proof This follows at once from (67), (68), and (95).

Of course $K^{l0}(x,y) = Z_{l,x}(y)$, the zonal harmonic in \mathcal{H}_{l0} based at x . It equals $[(2l+1)/4\pi]P_l^{1/2}(x \cdot y)$, where P_l^λ is the ultraspherical (or Gegenbauer) polynomial of degree l associated with $1/2$.

Series defining wavelets on the sphere, involving the K^{l0} , were estimated in [37], [38] and [18]. Using these estimates, and Theorem 5.6, we will be able to estimate similar series, which involve the K^{ls} for general s , in place of the K^{l0} , in the next section.

6 Spin Wavelets

In sections 3-5 we have often used the variable “ f ” to denote a spin function on the sphere. In the remainder of this article, the variable “ f ” will be reserved for another purpose.

Specifically, say $f \in \mathcal{S}(\mathbb{R}^+)$, $f \neq 0$, and $f(0) = 0$. Let (\mathbf{M}, g) be a smooth compact oriented Riemannian manifold, and let Δ be the Laplace-Beltrami operator on \mathbf{M} (for instance, the spherical Laplacian if \mathbf{M} is S^2). Let K_t be the kernel of $f(t^2\Delta)$. Then, as we have explained in section 2, the functions

$$w_{t,x}(y) = \overline{K}_t(x,y), \quad (104)$$

if multiplied by appropriate weights, can be used as wavelets on \mathbf{M} . In case $\mathbf{M} = S^n$ and f has compact support away from the origin, we shall say these wavelets are “needlet-type”.

On the sphere S^2 , we have

$$K_t(x,y) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f(t^2\lambda_l) Y_{lm}(x) \overline{Y_{lm}}(y). \quad (105)$$

As we have explained in section 2, a key property of this kind of wavelet is its *localization*. Working in the general situation (general \mathbf{M} , general f), one has:

for every pair of C^∞ differential operators X (in x) and Y (in y) on \mathbf{M} , and for every nonnegative integer N , there exists $c := c_{N,X,Y}$ such that for all $t > 0$ and $x, y \in \mathbf{M}$

$$|XYK_t(x,y)| \leq c \frac{t^{-(n+I+J)}}{(d(x,y)/t)^N}, \quad (106)$$

where $I := \deg X$ and $J := \deg Y$. In the case where \mathbf{M} is the sphere, f has compact support away from 0, and if one replaces $\lambda_l = l(l+1)$ by t^2 in the formula for K_t , this was shown by Narcowich, Petrushev and Ward, in [37] and [38]. The general result was shown in [18]. It will be important to note that in (106), the constant c can be chosen to be $\leq C\|f\|_M$, for some M . (Here we have chosen a nondecreasing family of norms $\{\|\cdot\|_M\}$ to define the Fréchet space topology on $\mathcal{S}_0(\mathbb{R}^+) = \{f \in \mathcal{S}(\mathbb{R}^+) : f(0) = 0\}$.) Indeed, this fact may be read off from the proof of (106). More simply, one may use the closed graph theorem: The collection of functions $K_t(x,y)$ (smooth in t, x, y for $t > 0$) which satisfy (106) is itself naturally a Fréchet space, say F . We want to see that the map taking f to the function $K_{t,f}(x,y) \in F$ is continuous. (Here we are temporarily letting $K_{t,f}$ be the kernel of $f(t^2\Delta)$.) For this, by the closed graph theorem, one need only note that the map from $\mathcal{S}_0(\mathbb{R}^+)$ to \mathbb{C} which takes f to $K_{t,f}(x,y)$ (for any fixed t, x, y) is continuous; but this is evident from (105).

The proof of (106) in [18] would in fact go through without change if, instead of K_t being the kernel of $f(t^2\Delta)$, where Δ is the Laplace-Beltrami operator, one assumed that K_t was the kernel of $f(t^2D)$, where D is any smooth positive second-order elliptic differential operator, acting on $C^\infty(\mathbf{M})$. It is natural to

conjecture then that something analogous holds if D is instead a smooth positive second-order elliptic differential operator, between sections of line bundles on \mathbf{M} .

In fact, the following theorem says that this is the case for $D = \Delta_s : C^\infty(\mathbf{L}^s) \rightarrow C^\infty(\mathbf{L}^s)$:

Theorem 6.1 *Let s be an integer. Suppose $f \in \mathcal{S}(\mathbb{R}^+)$, $f(0) = 0$. Let K_t be the kernel of $f(t^2 \Delta_s)$. Then:*

For every $R, R' \in SO(3)$, every pair of compact sets $\mathcal{F}_R \subseteq U_R$ and $\mathcal{F}_{R'} \subseteq U_{R'}$, and every pair of C^∞ differential operators X (in x) on $U_{R'}$ and Y (in y) on U_R , and for every nonnegative integer N , there exists c such that for all $t > 0$, all $x \in \mathcal{F}_{R'}$ and all $y \in \mathcal{F}_R$, we have

$$|XYK_{t,R',R}(x,y)| \leq c \frac{t^{-(n+I+J)}}{(d(x,y)/t)^N}, \quad (107)$$

where $I := \deg X$ and $J := \deg Y$.

Note The “needlet-type” case, where f has compact support away from the origin, is easiest, and that is the only case we will deal with in this article. The general case will be dealt with elsewhere.

Proof We have

$$f(t^2 \Delta_s) = \sum_{l \geq |s|} f(t^2 \lambda_{ls}) P_{ls} \quad (108)$$

(strong convergence), so that

$$K_{t,R',R}(x,y) = \sum_{l \geq |s|} f(t^2 \lambda_{ls}) K_{R',R}^{ls}. \quad (109)$$

(The interchange of order of integration and summation, needed to derive (109), is easily justified through the rapid decay of f , and through (71), once one recalls (95).) Now, say $s > 0$. Then by Theorem 5.6,

$$K_{t,R',R}(x,y) = \partial_{R'x}^{[s]} \bar{\partial}_{Ry}^{[s]} \sum_{l \geq |s|} f(t^2 \lambda_{ls}) \frac{(l-s)!}{(l+s)!} K^{l0}(x,y), \quad (110)$$

(The interchange of order of differentiation and summation, needed to derive (110), is easily justified through the rapid decay of f , and through (71) in the case $s = 0$, once one recalls (95) in the case $s = 0$.) Now notice that for $n \in \mathbb{Z}$, we have

$$(l+n)(l+1-n) = l(l+1) - \gamma_n \quad (111)$$

where

$$\gamma_n = n(n-1).$$

Also note that

$$\frac{(l+s)!}{(l-s)!} = [(l+s)(l+1-s)][[(l+s-1)(l+1-(s-1))]\dots[(l+1)(l+1-1)].$$

Accordingly

$$\frac{(l+s)!}{(l-s)!} = [l(l+1) - \gamma_s][l(l+1) - \gamma_{s-1}]\dots[l(l+1) - \gamma_1]. \quad (112)$$

Note also that

$$\lambda_{ls} = l(l+1) - \gamma_{-s} = l(l+1) - s(s+1). \quad (113)$$

To prove the theorem, let us first note that if we restrict to $t \in [T_0, T_1]$, where $0 < T_0 < T_1$ are fixed, then the result is trivial (even if we do not assume f has compact support away from the origin). Indeed, since $d(x,y) \leq 2\pi$ for all x, y , it is enough to show that $XYK_{t,R',R}$ is bounded for $(t, x, y) \in [T_0, T_1] \times \mathcal{F}_{R'} \times \mathcal{F}_R$. This, however, is evident from (71) and the rapid decay of f .

For other t , we shall assume that the support of f is contained in some interval $[a, b]$ with $b > a > 0$. In that case, there is a $T_1 > 0$ such that $K_{t,R,R'} \equiv 0$ for all $t > T_1$ (indeed, this will be true if $\lambda_{ss}T_1^2 > b$.) Thus we need only prove that there exists a $T_0 > 0$ such that the result holds for all $t \in (0, T_0]$. In fact, choose any $T_0 > 0$ with $\gamma_s T_0^2 < a/2$. For $0 < t \leq T_0$, define

$$f_t(u) = \frac{f(u - s(s+1)t^2)}{[u - \gamma_s t^2][u - \gamma_{s-1} t^2] \dots [u - \gamma_1 t^2]}. \quad (114)$$

The function f_t is supported in $[a, b + s(s+1)T_0^2]$, a fixed compact interval. (Note that the denominator in (114) does not vanish for u in this interval; in fact, each factor is at least $a/2$.) Then, in (110), by (112) we may write

$$\sum_{l \geq s} f(t^2 \lambda_{ls}) \frac{(l-s)!}{(l+s)!} K^{l0}(x, y) = t^{2s} \sum_{l \geq s} f_t(t^2 \lambda_l) K^{l0}(x, y) = t^{2s} K_{[t]}(x, y) \quad (115)$$

where $K_{[t]}$ is the kernel of $f_t(t^2 \Delta)$. Now, it is easy to see that the functions f_t ($t \in (0, T_0]$) form a bounded subset of $\mathcal{S}_0(\mathbb{R}^+)$. Thus, by the remarks following (106), for every pair of C^∞ differential operators X' (in x) and Y' (in y) on S^2 , and for every nonnegative integer N , there exists c such that whenever $0 < t \leq T_0$, and $x, y \in S^2$,

$$|X' Y' K_{[t]}(x, y)| \leq c \frac{t^{-(n+I'+J')}}{(d(x, y)/t)^N}, \quad (116)$$

where $I' := \deg X'$ and $J' := \deg Y'$. In fact, this remains true if we insist $x \in \mathcal{F}_{R'}$, $y \in \mathcal{F}_R$, and allow X' (resp. Y') to be smooth differential operators only in a neighborhood of $\mathcal{F}_{R'}$ (resp. \mathcal{F}_R), since there are smooth differential operators on all of S^2 which agree with these on $\mathcal{F}_{R'}$ (resp. \mathcal{F}_R). In the situation of (107), note that $\partial_{R'x}^{[s]}$ is a smooth differential operator of degree s in x in a neighborhood of $\mathcal{F}_{R'}$, and $\overline{\partial}_{R'y}^{[s]}$ is a smooth differential operator of degree s in y in a neighborhood of $\mathcal{F}_{R'}$. Thus we have

$$|XYK_{t,R',R}(x, y)| = t^{2s} \left| XY \partial_{R'x}^{[s]} \overline{\partial}_{R'y}^{[s]} K_{[t]}(x, y) \right| \leq ct^{2s} \frac{t^{-[n+(I+s)+(J+s)]}}{(d(x, y)/t)^N} = c \frac{t^{-[n+I+J]}}{(d(x, y)/t)^N},$$

as desired. Similar arguments work if $s < 0$. This completes the proof.

Remark 6.2 As in the case $s = 0$, the closed graph theorem implies that the constant c in (107) can be chosen to be $\leq C\|f\|_M$, for some M . (Here, for $f \in \mathcal{S}(\mathbb{R}^+)$, let us define $\|f\|_M = \sum_{i+j \leq M} \|x^i \partial^j f\|$, where $\|\cdot\|$ denotes sup norm.) Also, C and M depend only on I, J, N, R' and R .) Note, then, that (107) remains true if K_t is the kernel of $f(t^2 \Delta_s)$, where now only $f \in \mathcal{S}_0^M(\mathbb{R}^+) = \{f \in C^M(\mathbb{R}^+) : f(0) = 0 \text{ and } \|f\|_M < \infty\}$. This is evident, since $\mathcal{S}_0(\mathbb{R}^+) = \cap_L \mathcal{S}_0^L(\mathbb{R}^+)$ is dense in $\mathcal{S}_0^M(\mathbb{R}^+)$. Again we will can choose c to be $\leq C\|f\|_M$, where C and M depend only on I, J, N, R' and R .

Let us now explain what we shall mean by spin wavelets. Say $f \in \mathcal{S}(\mathbb{R}^+)$, $f \neq 0$, $f(0) = 0$. Say s is an integer, $R \in SO(3)$, $t > 0$, and $x \in U_R$. Let us define

$$w_{txR} = \sum_{l \geq |s|} \sum_m \overline{f}(t^2 \lambda_{ls}) \overline{s Y_{lmR}}(x) {}_s Y_{lm}. \quad (117)$$

Then $w_{txR} \in C^\infty(\mathbf{L}^s)$. Also note that

$$w_{txR, R'}(y) = \overline{K}_{t,R,R'}(x, y) \quad (118)$$

(if $x \in U_R$, $y \in U_{R'}$, and K_t is the kernel of $f(t^2 \Delta_s)$). This generalizes the case $s = 0$ of (104). Moreover, if $F \in L^2(\mathbf{L}^s)$, then for $x \in U_R$

$$\beta_{t,F,x,R} := \langle F, w_{txR} \rangle = (f(t^2 \Delta_s)F)_R(x) = (\beta_{t,F})_R(x) \quad (119)$$

where we have set $\beta_{t,F} = (f(t^2 \Delta_s)F)$. We call $\beta_{t,F,x,R}$ a *spin wavelet coefficient* of F . If $F = \sum_{l \geq |s|} \sum_m a_{lm} {}_s Y_{lm} \in L^2(\mathbf{L}^s)$, then

$$\beta_{t,F} = \sum_{l \geq |s|} \sum_m f(t^2 \lambda_{ls}) a_{lm} {}_s Y_{lm}(x). \quad (120)$$

so that

$$\beta_{t,F,x,R} = \sum_{l \geq |s|} \sum_m f(t^2 \lambda_{ls}) a_{lm} {}_s Y_{lmR}(x). \quad (121)$$

In cosmological applications, one assumes that f is *real-valued*. In that case, by (120) and (80) – (83), we have that

$$\beta_{t,F_E} = [\beta_t(F)]_E; \quad \beta_t(F_M) = [\beta_t(F)]_M \quad (122)$$

for all $F \in L^2(\mathbf{L}^s)$.

Let us now explain how, in analogy to the case $s = 0$, one can obtain a nearly tight frame from spin wavelets. Let $P = P_{|s|,s}$ be the projection onto the $\mathcal{H}_{|s|,s}$, the null space of Δ_{ls} (in $C^\infty(\mathbf{L}^s)$). Then for $a > 1$ sufficiently close to 1, for an appropriate discrete set $\{x_{j,k}\}_{(j,k) \in \mathbb{Z} \times [1, N_j]}$ on \mathbf{M} , certain $R_{j,k}$ with $x_{j,k} \in U_{R_{j,k}}$, and certain weights $\mu_{j,k}$, the collection of $\{\phi_{j,k} := \mu_{j,k} w_{a^j, x_{j,k}, R_{j,k}}\}$ constitutes a nearly tight wavelet frame for $(I - P)L^2(\mathbf{L}^s)$. (Note that this property is independent of the choice of $R_{j,k}$.) In that case, we will have that for some C , if $F \in (I - P)L^2(\mathbf{L}^s)$, then, in L^2 , $F \sim C \sum_{j,k} \langle F, \phi_{j,k} \rangle \phi_{j,k}$ (note that this sum is *independent* of the choice of $R_{j,k}$). We need to make this all precise.

In order to find suitable $x_{j,k}$, we need to subdivide S^2 into a fine grid. To this end, we select $c_0, \delta_0 > 0$ with the following properties:

(*) Whenever $0 < \tau < \delta$, we can write S^2 as a finite disjoint union of measurable sets \mathcal{E}_k such that the diameter of each \mathcal{E}_k is less than or equal to τ , and such that the measure of each \mathcal{E}_k is at least $c_0 \tau^2$.

It is easy to see that such τ, δ exist. (We could even require that each set \mathcal{E}_k be either a spherical cap, or a spherical rectangle bounded by latitude and longitude lines. The existence of c_0, δ as in (*) can even be established on general smooth compact oriented Riemannian manifolds; see the discussion before Theorem 2.4 of [19].)

By using Theorem 6.1, one can prove the following result, which generalizes Theorem 2.4 of [19] (when the manifold there is the sphere).

Theorem 6.3 (a) Fix $a > 1$, and say c_0, δ_0 are as in (*) above. Suppose $f \in \mathcal{S}(\mathbb{R}^+)$, and $f(0) = 0$. Let K_t be the kernel of $f(t^2 \Delta_s)$, and let w_{txR} be as in (117) and (118) above.

Then there exists a constant $C_0 > 0$ (depending only on f, a, c_0 and δ_0) as follows:

Let $\mathcal{J} \subseteq \mathbb{Z}$ be either finite, or equal to all of \mathbb{Z} . Say $0 < b < 1$. For each $j \in \mathcal{J}$, write S^2 as a finite disjoint union of measurable sets $\{E_{j,k} : 1 \leq k \leq N_j\}$, where:

$$\text{the diameter of each } E_{j,k} \text{ is less than or equal to } ba^j, \quad (123)$$

and where:

$$\text{for each } j \text{ with } ba^j < \delta_0, \mu(E_{j,k}) \geq c_0(ba^j)^2. \quad (124)$$

For each j, k , select $x_{j,k} \in E_{j,k}$ and select any $R_{j,k} \in SO(3)$ with $x_{j,k} \in U_{R_{j,k}}$. Set

$$\varphi_{j,k} = w_{a^j, x_{j,k}, R_{j,k}} \in C^\infty(\mathbf{L}^s). \quad (125)$$

For $F \in L^2(\mathbf{L}^s)$, set

$$S^{\mathcal{J}} F := \sum_{j \in \mathcal{J}} \sum_k \mu(E_{j,k}) \langle F, \varphi_{j,k} \rangle \varphi_{j,k}.$$

Here, if $\mathcal{J} = \mathbb{Z}$, the series converges unconditionally in $L^2(\mathbf{L}^s)$.

Let $Q^{\mathcal{J}} = \sum_{j \in \mathcal{J}} |f|^2 (a^{2j} \Delta_s)$; if $\mathcal{J} = \mathbb{Z}$, this series converges strongly on $L^2(\mathbf{L}^s)$. Then for all $F \in L^2(\mathbf{L}^s)$,

$$|\langle (Q^{\mathcal{J}} - S^{\mathcal{J}})F, F \rangle| \leq C_0 b \langle F, F \rangle \quad (126)$$

(or, equivalently, since $Q^{\mathcal{J}} - S^{\mathcal{J}}$ is self-adjoint, $\|Q^{\mathcal{J}} - S^{\mathcal{J}}\| \leq C_0 b$.)

(b) In (a), take $\mathcal{J} = \mathbb{Z}$; set $Q = Q^{\mathbb{Z}}$, $S = S^{\mathbb{Z}}$. Suppose that the Daubechies condition (5) holds. Let $P = P_{|s|, s}$ be the projection onto the $\mathcal{H}_{|s|, s}$, the null space of Δ_s (in $C^\infty(\mathbf{L}^s)$). Then

$$(A_a - C_0 b)(I - P) \leq S \leq (B_a + C_0 b)(I - P) \quad (127)$$

as operators on $L^2(\mathbf{L}^s)$. Thus, for any $F \in (I - P)L^2(\mathbf{M})$,

$$(A_a - C_0 b)\|F\|^2 \leq \sum_{j,k} \mu(E_{j,k}) |\langle F, \varphi_{j,k} \rangle|^2 \leq (B_a + C_0 b)\|F\|^2,$$

so that, if $A_a - C_0 b > 0$, then $\{\mu(E_{j,k})^{1/2} \varphi_{j,k}\}_{j,k}$ is a frame for $(I - P)L^2(\mathbf{L}^s)$, with frame bounds $A_a - C_0 b$ and $B_a + C_0 b$.

Remarks 1. By (6) and (7), $B_a/A_a = 1 + O(|(a-1)^2(\log|a-1|)|)$; evidently $(B_a + C_0 b)/(A_a - C_0 b)$ can be made arbitrarily close to B_a/A_a by choosing b sufficiently small. Thus, Theorem 6.3 gives “nearly tight” frames for $(I - P)L^2(\mathbf{L}^s)$.

2. The proof of Theorem 6.3 follows along the same lines of the proof for $s = 0$, which is Theorem 2.4 of [19]. It will be given elsewhere [21]. In this article, we will only use Theorem 6.3 in Section 8.

3. In the statement of Theorem 2.4 (a) of [19], \mathcal{J} was not assumed to be a finite set, but rather, a cofinite set (that is, \mathcal{J}^c was assumed to be finite), and one had unconditional convergence of the sum for $S^{\mathcal{J}}F$ and strong convergence of the sum for $Q^{\mathcal{J}}$. Note, however, that this is equivalent to (a) for finite sets, since $Q^{\mathcal{J}} = Q^{\mathbb{Z}} - Q^{\mathcal{J}^c}$, $S^{\mathcal{J}}F = S^{\mathbb{Z}}F - S^{\mathcal{J}^c}F$.

7 Asymptotic Uncorrelation

We define a random spin s field $G(\cdot)$ by assuming that there exists a probability space $(\Omega, \mathfrak{F}, P)$ such that the map $(x, \omega) \rightarrow G(x, \omega)$ is $\mathcal{B}(S^2) \otimes \mathfrak{F}$ measurable, \mathfrak{F} denoting a σ -algebra on Ω and $\mathcal{B}(S^2)$ the Borel σ -field of S^2 . (To clarify: $G(x, \omega)$ is a.e. in \mathbf{L}_x^s (the fiber of \mathbf{L}^s above x), and to say that the map is measurable is to say that, for any $R \in SO(3)$ the complex-valued function over $U_R \times \Omega$ obtained from the map by trivializing the bundle over U_R is measurable.) As is customary, the dependence of G upon ω will be suppressed in the notation. Also, we assume that

$$E \left[\int_{S^2} |G(x)|^2 dS(x) \right] = C < \infty ,$$

which in particular entails that $x \rightarrow G(x)$ belongs to $L^2(\mathbf{L}^s)$ with probability one (see for instance [40]). In cosmological applications it is natural to introduce an *isotropy condition*:

Definition 7.1 Let G be a random spin s field. We say that G is isotropic (in law) if for every $x_1, \dots, x_N \in U_I$, the joint probability distribution of $G_I^R(x_1), \dots, G_I^R(x_N)$ is independent of $R \in SO(3)$. Similarly, if G, H are both random spin s fields defined on the same probability space, we say that G, H are jointly isotropic (in law) if for every $x_1, \dots, x_N, y_1, \dots, y_M \in U_I$, the joint probability distribution of $G_I^R(x_1), \dots, G_I^R(x_N), H_I^R(y_1), \dots, H_I^R(y_M)$ is independent of $R \in SO(3)$.

Note that G is isotropic if and only if, for every $R' \in SO(3)$, and for every $p_1, \dots, p_N \in U_{R'}$, the joint probability distribution of $G_{R'}^R(p_1), \dots, G_{R'}^R(p_N)$ is independent of $R \in SO(3)$. Indeed, say $p_j = R'x_j$, $x_j \in U_I$ (for $j = 1, \dots, N$). One need only then observe that, by (28), $G_{R'}^R(p_j) = G_I^{RR'}(x_j)$ for each j . Similarly, G, H are jointly isotropic if and only if, for every $R' \in SO(3)$, and for every $p_1, \dots, p_N, q_1, \dots, q_M \in U_{R'}$,

the joint probability distribution of $G_{R'}^R(p_1), \dots, G_{R'}^R(p_N), G_{R'}^R(q_1), \dots, G_{R'}^R(q_N)$ is independent of $R \in SO(3)$.

If G is isotropic, then for any $R' \in SO(3)$ and any $p \in U_{R'}$, the probability distribution of $G_{R'}^R(p)$ is independent of $R \in SO(3)$. Restricting R to the subgroup of $SO(3)$ which fixes p , we see in particular that $G_{R'}^R(p)$ has the same probability distribution as $G_{R'R}(p) = e^{is\psi}G_{R'}^R(p)$, where ψ is the angle from $\rho_{R'}(p)$ to $\rho_{R'R}(p)$. If $s \neq 0$, we may take ψ with $e^{is\psi} = -1$. We then see that the probability distributions of $\Re G_{R'}^R(p)$ is the same as that of $-\Re G_{R'}^R(p)$, similarly for $\Im G_{R'}^R(p)$; in particular, both $\Re G_{R'}^R(p)$ and $\Im G_{R'}^R(p)$ must have expectation 0. Again if $s \neq 0$, we may take ψ with $e^{is\psi} = i$, and we see now that $\Re G_{R'}^R(p)$ and $\Im G_{R'}^R(p)$ have the same probability distributions.

Say G is isotropic, and that G^R has spin s spherical harmonic coefficients a_{lm}^R (for $R \in SO(3)$). Then, evidently, the a_{lm}^R all have the same probability distribution, and each a_{lm}^R has expectation zero. (Here $a_{lm}^R = \langle G^R, {}_s Y_{lm} \rangle = \langle G_{R'}^R, {}_s Y_{lmR'} \rangle$ (for any $R' \in SO(3)$) where $\langle \cdot \rangle$ is the inner product on S^2 .)

If we take R to be a rotation about \mathbf{N} , we see (from $a_{lm}^R = \langle G, {}_s Y_{lm}^{R^{-1}} \rangle$) and from (60) that, if $m \neq 0$, then the probability distribution of a_{lm} is the same as that of $e^{i\theta}a_{lm}$ for any θ . In particular, $\Re a_{lm}$ and $\Im a_{lm}$ will have the same probability distributions, if $m \neq 0$.

If G, H are jointly isotropic, $a_{lm}^R = \langle G^R, {}_s Y_{lm} \rangle$, $b_{lm}^R = \langle H^R, {}_s Y_{lm} \rangle$, then for any l, m, l', m' the covariances $E(a_{lm}^R \bar{b}_{l'm'}^R)$ are evidently independent of R .

In fact, as in the case $s = 0$ of [6], we can say much more about these covariances:

Theorem 7.2 Suppose G, H are jointly isotropic random spin s fields. Let $a_{lm} = \langle G^R, {}_s Y_{lm} \rangle$, and $b_{lm} = \langle H^R, {}_s Y_{lm} \rangle$. Then $E(a_{lm} \bar{b}_{l'm'}) = 0$ unless $l = l'$ and $m = m'$. Moreover $E(a_{lm} \bar{b}_{lm})$ does not depend on m ; we will denote it by $C_{l,G,H}$.

Proof We define Wigner's D^l matrices by

$${}_s Y_{lm}^R = \sum_m D_{m'm}^l(R) {}_s Y_{lm'} . \quad (128)$$

These matrices surely exist and are unitary, since the ${}_s Y_{lm}^R$ form an orthonormal basis of \mathcal{H}_{ls} whenever $R \in SO(3)$ and $l \geq |s|$. The matrices are independent of s , as one sees at once from (128), use of the ∂ and $\bar{\partial}$ operators, and (67), (68), (40), (55). Since ${}_s Y_{lm}^{RR'} = \sum_m D_{m'm}^l(R) {}_s Y_{lm'}^{R'}$ for any R, R' , we see that in fact the map $R \rightarrow D_{m'm}^l(R)$ is a unitary representation of $SO(3)$, see also [45] and [48]. This representation is evidently isomorphic to the action of $SO(3)$ on \mathcal{H}_{l0} , which as is well known, is irreducible; moreover, the representations for different l are not equivalent.

Now let $a_{lm}^R = \langle G^R, {}_s Y_{lm} \rangle$, and $b_{lm}^R = \langle H^R, {}_s Y_{lm} \rangle$. For each l, l' define the covariance matrix $M^{ll'}$ by $M_{mm'}^{ll'} = E(a_{lm} \bar{b}_{l'm'}) = E(a_{lm}^R \bar{b}_{l'm'}^R)$ for any R . Noting that $a_{lm}^R = \langle G, {}_s Y_{lm}^{R^{-1}} \rangle$, one easily computes that $M^{ll'} = [D^l(R)]^* M^{ll'} D^l(R)$ for any R . The theorem now follows at once from Schur's lemma.

In cosmological applications, having an “asymptotic uncorrelation” theorem of the following kind is of great importance. Generalizing a key result in [4], we now show:

Theorem 7.3 Let G, H be random, jointly isotropic spin s fields. If $x \in U_R$, let w_{txR} be the spin wavelet of (117), where f has compact support away from the origin. Assume that $C_{l,G,H} = g(l)$ for a smooth function g on the interval $(|s|, \infty)$, which satisfies the following condition, for some $\alpha > 2$: for every $i \in \mathbb{N}_0$ there exists $c_i > 0$ such that

$$|g^{(i)}(u)| \leq c_i u^{-\alpha-i}. \quad (129)$$

Assume also that, for some $c > 0$, certain $\rho, \tau \in \mathbb{R}$, and for all sufficiently large l ,

$$C_{l,G,G} \geq cl^{-\rho}, \text{ and } C_{l,H,H} \geq cl^{-\tau}. \quad (130)$$

For $x \in U_R$, let $\beta_{G,t,x,R} = \langle G, w_{txR} \rangle$, $\beta_{H,t,x,R} = \langle H, w_{txR} \rangle$. Then for any $M \in \mathbb{N}_0$, there exists $T_0 > 0$ and a constant $C_M > 0$, such that for all $x, y \in U_R$, $0 < t < T_0$, we have

$$|\text{Cor}(\beta_{G,t,x,R}, \beta_{H,t,y,R})| \leq t^{\alpha-(\rho+\tau)/2} \frac{C_M}{(d(x, y)/t)^M}. \quad (131)$$

In particular, for fixed x, y , $|\text{Cor}(\beta_{G,t,x,R}, \beta_{H,t,y,R})| \rightarrow 0$ as $t \rightarrow 0^+$.

Remark: Note that $|\text{Cor}(\beta_{t,x,R}, \beta_{t,y,R})|$ is independent of R .

Proof By definition,

$$|\text{Cor}(\beta_{G,t,x,R}, \beta_{H,t,y,R})| = \frac{|E(\beta_{G,t,x,R} \overline{\beta}_{H,t,y,R})|}{\sqrt{E(|\beta_{G,t,y,R}|^2)} \sqrt{E(|\beta_{H,t,y,R}|^2)}}. \quad (132)$$

Say G, H have spin s spherical harmonic coefficients a_{lm}, b_{lm} respectively. Note that

$$\beta_{G,t,x,R} = \sum_{l \geq |s|} \sum_m f(t^2 \lambda_{ls}) a_{lm} {}_s Y_{lmR}(x), \quad (133)$$

and similarly for H . Accordingly

$$E(\beta_{G,t,x,R} \overline{\beta}_{H,t,y,R}) = \sum_{l \geq |s|} |f|^2(t^2 \lambda_{ls}) C_{l,G,H} \sum_m {}_s Y_{lmR}(x) \overline{{}_s Y_{lmR}}(y) \quad (134)$$

$$= \sum_{l \geq |s|} |f|^2(t^2 \lambda_{ls}) C_{l,G,H} K_{R,R}^{ls}(x, y). \quad (135)$$

To estimate $E(|\beta_{G,t,x,R}|^2)$, let us note that, since $f \neq 0$, we may choose c_0 and $0 < a < b$ such that $|f|^2 \geq c_0$ on $[a^2, b^2]$. Thus $|f|^2(t^2 \lambda_{ls}) \geq c_0$ if $a/t \leq \sqrt{\lambda_{ls}} \leq b/t$. Choose a', b' with $a < a' < b' < b$. Since $\lim_{l \rightarrow \infty} l/\sqrt{\lambda_{ls}} = 1$, there is a $T_1 > 0$ such that if $0 < t < T_1$, $|f|^2(t^2 \lambda_{ls}) \geq c_0$ if $a'/t \leq l \leq b'/t$. Choose $0 < T_2 < T_1$ with $(b' - a')/t > 1$ if $0 < t < T_2$. Then, for t sufficiently small, by Theorem 5.5 and (130), for some $C', C > 0$,

$$E(|\beta_{G,t,x,R}|^2) \geq \sum_{\frac{a'}{t} \leq l \leq \frac{b'}{t}} c_0 l^{-\rho} \frac{2l+1}{4\pi} \geq C' \frac{b' - a'}{t} t^\alpha t^{-1} = Ct^{-2+\rho}. \quad (136)$$

Similarly,

$$E(|\beta_{H,t,x,R}|^2) \geq Ct^{-2+\tau}. \quad (137)$$

We estimate the numerator in (132) by using Theorem 6.1. In order to do this, we must transform g in some simple ways.

Say $s \geq 0$. Recall that $\lambda_{ls} = (l-s)(l+s+1) = l(l+1) - s(s+1)$. Thus, if $v > 0$, $v = \lambda_{ls}$ if and only if $l = \frac{1}{2}[-1 + \sqrt{1 + 4[v + s(s+1)]}] := Q(v)$, say. Then $Q : (0, \infty) \rightarrow (0, \infty)$ bijectively. For $v > 0$, set $h(v) = g(Q(v))$; then $g(l) = h(\lambda_{ls})$. Moreover, it is easy to see, from (129), that for every $i \in \mathbb{N}_0$ there exists $C'_i > 0$ such that

$$|\frac{d^i}{dv^i} h(v)| \leq C'_i v^{-\alpha/2-i} \quad (138)$$

From (135), we have

$$E(\beta_{txR} \overline{\beta}_{tyR}) = \sum_{l \geq |s|} |f|^2(t^2 \lambda_{ls}) h(\lambda_{ls}) K_{R,R}^{ls}(x, y) \quad (139)$$

$$= t^\alpha K_{t,R,R}(x, y), \quad (140)$$

where K_t is the kernel of $f_{[t]}(t^2 \Delta_s)$, where now

$$f_{[t]}(w) = |f|^2(w)[t^{-\alpha}h(w/t^2)], \quad (141)$$

(since $|f|^2(t^2 \lambda_{ls})h(\lambda_{ls}) = t^\alpha f_{[t]}(t^2 \lambda_{ls})$). Select $0 < A < B$ with $\text{supp } f \subseteq [A, B]$. Then $\text{supp } f_{[t]} \subseteq [A, B]$ for all t . By use of (138), one sees readily that for any $L \in \mathbb{N}_0$, there is a $C > 0$ such that $\|f_{[t]}\|_{C^L} \leq C$ for all t . Accordingly, by Theorem 6.1 and Remark 6.2, for every M there exists $C'_M > 0$ with

$$|E(\beta_{txR} \overline{\beta}_{tyR})| \leq \frac{C'_M t^{\alpha} t^{-2}}{(d(x, y)/t)^M}$$

Using this in (132) together with (136), we find the desired result. Similarly for $s < 0$.

Remarks 1. Since $\text{supp } f_{[t]} \subseteq [A, B]$, when deriving the estimate $\|f_{[t]}\|_{C^L} \leq C$ from (138) for t small, one need only use (138) for v large (by (141)). Thus, instead of assuming g is smooth on $(|s|, \infty)$, and that (129) holds there, one may assume only that for some $T > 0$, g is smooth on (T, ∞) , and that (129) holds there. Equivalently, we are assuming that $g(u)$ agrees with an ordinary symbol of order $-\alpha$ for large u .

2. By remark (6.2), an examination of the proof of the theorem shows that for each M there is an $L \in \mathbb{N}_0$ such that, instead of assuming that g is smooth and satisfies (129) for all i , we need only assume that $g \in C^L$ and satisfies (129) for all $i \leq L$.

3. The hypothesis (129), which is used only to estimate the derivatives of h and then the derivatives of $f_{[t]}$, can evidently be relaxed – at a cost – to $|g^{(i)}(u)| \leq c_i u^{-\alpha_i}$ whenever $i \leq L$, for certain α_i . The cost in doing this is that one would need to multiply the right side of (131) by a factor of t^{-N} for some sufficiently large N . It would still evidently follow that for fixed x, y , $|\text{Cor}(\beta_{G,t,x,R}, \beta_{H,t,y,R})| \rightarrow 0$ as $t \rightarrow 0^+$.

4. If $G = H$, in applications of the theorem, it is natural to assume $\alpha = \rho = \tau$.

5. In cosmological applications, where we have (122), and where $s = 2$, one assumes that the polarization field is (a single sample of) an isotropic random field F , and that F_E and F_M are jointly isotropic.

8 Stochastic Limit Theorems

In this final section, we give a brief glimpse of the significance of the results of the previous section in cosmology. We prove some stochastic limit results, and then indicate why they are important.

Say $\epsilon > 0$.

We use a nearly tight frame, as provided by Theorem 6.3. Specifically, in that theorem, we fix a real f supported in $[1/a^2, a^2]$ for which the Daubechies sum $\sum_{j=-\infty}^{\infty} f^2(a^{2j}u) = 1$ for all $u > 0$, so that $A_a = B_a = 1$. We then choose b sufficiently small that $C_0 b < \epsilon$. We then produce $\{\phi_{j,k}\}$ as in Theorem 6.3, and let $\phi_{j,k} = \mu(E_{j,k})^{1/2} \varphi_{j,k}$, so that $\{\phi_{j,k}\}$ is a nearly tight frame for $(I - P)L^2(\mathbf{L}^s)$, with frame bounds $1 - \epsilon$ and $1 + \epsilon$. Also, we have (126) for general finite subsets \mathcal{J} . Of course our choice of $\{\phi_{j,k}\}$ depends on ϵ ; if we want to indicate the dependence on ϵ , we will write $\phi_{j,k,\epsilon}$.

Say $G = \sum A_{lm} s Y_{lm}$ is an isotropic random spin s field, and let $C_l = C_{l,G,G}$ (notation as in Theorem 7.3). For $j \in \mathbb{Z}$, we also let

$$\gamma_j = \sum_{a^{-2} \leq a^{2j} \lambda_{ls} \leq a^2} C_l(2l+1). \quad (142)$$

(To avoid confusion, the summation in (142), and similar summations in the sequel, are over those l for which λ_{ls} is defined and for which $a^{-2} \leq a^{2j} \lambda_{ls} \leq a^2$; recall that λ_{ls} is defined for $l \geq |s|$.) Note that, for any l , there at most 3 values of j for which $a^{-2} \leq a^{2j} \lambda_{ls} \leq a^2$, so that

$$\sum_j \gamma_j \leq 3 \sum_{l \geq |s|} C_l(2l+1) = 3\text{Var}G. \quad (143)$$

We let $\beta_{jk} = \beta_{jk\epsilon} = \langle G, \phi_{j,k,\epsilon} \rangle$. We focus on the quadratic statistics

$$\tilde{\Gamma}_j = \sum_k |\beta_{jk}|^2 \quad (144)$$

and

$$\hat{\Gamma}_j = \sum_{l \geq |s|, m} f^2(a^{2j} \lambda_{ls}) |A_{lm}|^2. \quad (145)$$

Evidently

$$E(\hat{\Gamma}_j) = \sum_{l \geq |s|} f^2(a^{2j} \lambda_{ls}) C_l(2l+1) = \sum_{a^{-2} \leq a^{2j} \lambda_{ls} \leq a^2} f^2(a^{2j} \lambda_{ls}) C_l(2l+1). \quad (146)$$

We have:

Proposition 8.1 $E(|\hat{\Gamma}_j - \tilde{\Gamma}_j|) \leq \epsilon \gamma_j$ for all j .

Proof For each $j \in \mathbb{Z}$, let

$$G_j = \sum_{a^{-2} \leq a^{2j} \lambda_{ls} \leq a^2} A_{lm} s Y_{lm}. \quad (147)$$

Then

$$\beta_{jk} = \langle G, \phi_{j,k} \rangle = \mu(E_{j,k})^{1/2} \sum_{l \geq |s|} \sum_m f(a^{2j} \lambda_{ls}) A_{lm} s Y_{lm} = \langle G_j, \phi_{j,k} \rangle.$$

Note also that $(I - P)G_j = G_j$, since no term with $l = |s|$ can appear in the summation in (147) (for then $\lambda_{ls} = 0$). By Theorem 6.3 (a) with $\mathcal{J} = \{j\}$, we see that, in the notation of that theorem,

$$\tilde{\Gamma}_j = \langle S^{\mathcal{J}} G_j, G_j \rangle,$$

$$\hat{\Gamma}_j = \langle Q^{\mathcal{J}} G_j, G_j \rangle,$$

so that, by Theorem 6.3 (a),

$$|\hat{\Gamma}_j - \tilde{\Gamma}_j| \leq \epsilon \|G_j\|_2^2 = \epsilon \sum_{a^{-2} \leq a^{2j} \lambda_{ls} \leq a^2} |A_{lm}|^2.$$

The proposition now follows at once if we take expectations of both sides.

In Proposition 8.1, $\tilde{\Gamma}_j = \sum_k |\beta_{jk\epsilon}|^2$ depends on our choice of ϵ , and we can even let $\epsilon \ll 1$ depend on j here. By Tchebychev's inequality, the probability distribution of $\tilde{\Gamma}_j$ is then a small perturbation of that of $\hat{\Gamma}_j$. We conclude by briefly discussing the latter.

We will look at random spin s fields $G = \sum A_{lm} s Y_{lm}$ for which $\bar{A}_{lm} = A_{l,-m}$ for all m (so that, in particular, A_{l0} is real). For want of a better word, let us call such a G *involutive*. Note that, by (80) – (83), if F is a random spin s field, then both $F_{\mathbf{E}}$ and $F_{\mathbf{M}}$ are involutive.

We will also be assuming that G is *Gaussian*, by which we mean that $\{\Re G_I(x) : x \in U_I\} \cup \{\Im G_I(x) : x \in U_I\}$ is a Gaussian family. In that case $\mathcal{F} := \{\Re A_{lm}\} \cup \{\Im A_{lm}\}$ is also a Gaussian family. Recall that, if $m \neq 0$, $\{\Re A_{lm}\}$ and $\{\Im A_{lm}\}$ have the same probability distributions, so that they have variances $E|\Re A_{lm}|^2$, $E|\Im A_{lm}|^2 = C_l/2$. Since we are assuming G is involutive, $\Re A_{lm} = (A_{lm} + A_{l,-m})/2$, $\Im A_{lm} = -i(A_{lm} - A_{l,-m})/2$. It follows easily from this and from Theorem 7.2 that the elements of \mathcal{F} are pairwise uncorrelated. Since \mathcal{F} is a Gaussian family they are independent.

We then have:

Proposition 8.2 For all $j = 1, 2, \dots,$,

$$\text{Var} \left\{ \widehat{\Gamma}_j \right\} = E \left[\widehat{\Gamma}_j - E \widehat{\Gamma}_j \right]^2 = \sum_{l \geq |s|} 2f^4(a^{2j}\lambda_{ls})(C_l)^2(2l+1) .$$

Say now that for some $c, C > 0$ and some $\alpha > 2$, $cl^{-\alpha} \leq C_l \leq Cl^{-\alpha}$ for $l > 0$. Then

$$\frac{\widehat{\Gamma}_j - E \widehat{\Gamma}_j}{\sqrt{\text{Var} \left\{ \widehat{\Gamma}_j \right\}}} \xrightarrow{d} N(0, 1) \text{ as } j \rightarrow -\infty ,$$

\xrightarrow{d} denoting as usual convergence in probability law, and $N(0, 1)$ denoting as usual the standard normal distribution.

Proof Write

$$X_{lm}(j) := \begin{cases} \sqrt{2}f(a^{2j}\lambda_{ls})\Re A_{lm} & \text{for } m > 0 \\ f(a^{2j}\lambda_{ls})A_{l0} & \text{for } m = 0 \\ \sqrt{2}f(a^{2j}\lambda_{ls})\Im A_{lm} & \text{for } m < 0. \end{cases} ;$$

For each j , $\{X_{lm}(j)\}$ is a triangular array of independent heteroscedastic ($=$ unequal variance) Gaussian random variables such that

$$\begin{aligned} \widehat{\Gamma}_j &= \sum_{l \geq |s|, m} X_{lm}^2(j) , \quad EX_{lm}^2(j) = f^2(a^{2j}\lambda_{ls})C_l , \\ \text{Var} \left\{ X_{lm}^2(j) \right\} &= EX_{lm}^4(j) - \{EX_{lm}^2(j)\}^2 = 2f^4(a^{2j}\lambda_{ls})(C_l)^2 . \end{aligned}$$

Hence we have easily

$$\begin{aligned} \text{Var} \left\{ \widehat{\Gamma}_j \right\} &= \sum_{l \geq |s|, m} \text{Var} \left\{ X_{lm}^2(j) \right\} = \sum_{l \geq |s|, m} 2f^4(a^{2j}\lambda_{ls})(C_l)^2 \\ &= \sum_{l \geq |s|} 2f^4(a^{2j}\lambda_{ls})(C_l)^2(2l+1) . \end{aligned}$$

To establish the Central Limit Theorem, it is then enough to check the validity of the Lindeberg-Levy condition ([10]) which here takes the simple form

$$\begin{aligned} &\lim_{j \rightarrow -\infty} \frac{\max_{a^{-2} \leq a^{2j}\lambda_{ls} \leq a^2} \max_m \text{Var}(X_{lm}^2)}{\sum_{a^{-2} \leq a^{2j}\lambda_{ls} \leq a^2} 2f^4(a^{2j}\lambda_{ls})(C_l)^2} \\ &\leq K \lim_{j \rightarrow -\infty} \frac{\max_{a^{-2} \leq a^{2j}\lambda_{ls} \leq a^2} \max_m f^4(a^{2j}\lambda_{ls})(C_l)^2}{\sum_{a^{-2} \leq a^{2j}\lambda_{ls} \leq a^2} f^4(a^{2j}\lambda_{ls})(C_l)^2(2l+1)} \\ &\leq K \lim_{j \rightarrow -\infty} \frac{a^{2j\alpha}}{a^{2j\alpha} \sum_{a^{-2} \leq a^{2j}\lambda_{ls} \leq a^2} f^4(a^{2j}\lambda_{ls})l} = 0 . \end{aligned}$$

This completes the proof.

References

- [1] Antoine, J.-P. and Vandergheynst, P. (1999) *Wavelets on the Sphere: a Group-Theoretic Approach, Applied and Computational Harmonic Analysis*, 7, pp. 262-291
- [2] Antoine J.-P., Demanet L., Jacques L., Vandergheynst P. (2002) *Wavelets on the sphere: implementation and approximations, Applied and Computational Harmonic Analysis*, 13, 177–200.

- [3] Antoine, J.-P. and Vandergheynst, P. (2007), *Wavelets on the Sphere and Other Conic Sections*, Journal of Fourier Analysis and its Applications, **13**, 369-386
- [4] Baldi, P., Kerkyacharian, G., Marinucci, D. and Picard, D. *Asymptotics for Spherical Needlets*, Annals of Statistics, in press, arxiv:math/0606599.
- [5] Baldi, P., Kerkyacharian, G. Marinucci, D. and Picard, D. (2007) *Subsampling Needlet Coefficients on the Sphere*, Bernoulli, in press, arxiv 0706.4169
- [6] Baldi, P., Marinucci, D. and Varadarajan, V.S., *On the Characterization of Isotropic Gaussian Fields on Homogeneous Spaces of Compact groups*, Electronic Communications in Probability, (2007), pp. 291-302
- [7] Cabella, P. and Kamionkowski, M. *Theory of cosmic microwave background polarization*, arXiv:astro-ph/0403392v2 18 Mar 2005
- [8] Cabella, P., Hansen, F.K., Marinucci, D., Pagano, D. and Vittorio, N. *Search for non-Gaussianity in Pixel, Harmonic, and Wavelet Space: Compared and Combined*. Physical Review D, **69** (2004) 063007.
- [9] Cruz, M., Cayon, L., Martinez-Gonzalez, E., Vielva, P., Jin, J., *The non-Gaussian Cold Spot in the 3-year WMAP Data*, Astrophysical Journal (2007) 655, 11-20
- [10] Davidson, J. *Stochastic Limit Theory*, (1994), Cambridge University Press
- [11] Delabrouille, J., Cardoso, J.-F. , Le Jeune, M. , Betoule, M., Fay, G. ,Guilloux, F. *A full sky, low foreground, high resolution CMB map from WMAP*, (2008), arxiv 0807.0773
- [12] Dodelson, S. *Modern Cosmology*, (2003) Academic Press
- [13] Eastwood, M. and Tod, P. *Edth – a differential operator on the sphere*, Math. Proc. Camb. Phil. Soc. **92** (1982), 317-330.
- [14] Fay, G., Guilloux, F. , Betoule, M. , Cardoso, J.-F. , Delabrouille, J. , Le Jeune, M. *CMB power spectrum estimation using wavelets*, (2008) arxiv 0807.1113
- [15] Fay, G. and Guilloux, F. *Consistency of a needlet spectral estimator on the sphere*, (2008) arxiv: 0807.2162
- [16] Geller, D. and Mayeli, A. *Continuous wavelets and frames on stratified Lie groups I*, Journal of Fourier Analysis and Applications, **12** (2006), 543-579.
- [17] Geller, D., Hansen, F.K., Marinucci, D., Kerkyacharian, G. and Picard, D., *Spin needlets for cosmic microwave background polarization data analysis*, preprint, 31 pages, on ArXiv (Physics section).
- [18] Geller, D. and Mayeli, A. *Continuous wavelets on manifolds*, accepted by and published on the website of Math. Z., 33 pages, also on ArXiv.
- [19] Geller, D. and Mayeli, A. *Nearly tight frames and space-frequency analysis on compact manifolds*, accepted by and published on the website of Math. Z., 30 pages, also on ArXiv.
- [20] Geller, D. and Mayeli, A. *Besov spaces and frames on compact manifolds*, arXiv:0709.2452 (September 2007).
- [21] Geller, D. and Mayeli, A. *Nearly tight frames for spin wavelets on the sphere*, in preparation.
- [22] Goldberg, J.N., Macfarlane, A. J. , Newman, E. T. , Rohrlich F. and Sudarshan, E. C. G. *Spin-s Spherical Harmonics and ∂* , J. Math. Phys. **8** (1967), 2155–2161.

- [23] Gorski, K. M., Lilje P. B. , *Foreground Subtraction of Cosmic Microwave Background Maps using WI-FIT (Wavelet based hIgh resolution Fitting of Internal Templates)*, *Astrophysical J.*, 648, (2006), 784–796.
- [24] Gorski, K.M. , Hivon, E. , Banday, A.J., Wandelt, B.D. Hansen, F.K., Reinecke, M. ,Bartelman, M. *HEALPix – a framework for high resolution discretization, and fast analysis of data distributed on the sphere*, *Astrophysical Journal* **622** (2005), 759-771.
- [25] Guilloux, F., Fay, G., Cardoso, J.-F. *Practical Wavelet Design on the Sphere*, (2007) arxiv 0706.2598
- [26] Hörmander, L. *Introduction to Complex Analysis in Several Variables*, (1973), North Holland.
- [27] Hinshaw, G., Weiland, J. L., Hill, R. S., Odegard, N., Larson, D., Bennett, C. L., Dunkley, J., Gold, B., Greason, M. R., Jarosik, N., Komatsu, E., Nolta, M. R., Page, L., Spergel, D. N., Wollack, E., Halpern, M., Kogut, A., Limon, M., Meyer, S. S., Tucker, G. S., Wright, E. L. (2008) *Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Data Processing, Sky Maps, and Basic Results* preprint arXiv:0803.0732
- [28] Kamionkowski, M., Kosowsky, A., Stebbins, A. *Statistics of cosmic microwave background polarization*, *Physical Review D*, Volume 55, Issue 12, pp.7368-7388
- [29] Lan, X. and D. Marinucci *The needlets bispectrum*, *Electronic Journal of Statistics*, Vol. 2 (2008), pp. 332-367
- [30] Lan, X. and D. Marinucci, *On the dependence structure of wavelet coefficients for spherical random fields*, preprint (2008), arXiv:0805.4154.
- [31] Laureijs, R. J.(On Behalf Of The Planck Collaboration), *Polarization Maps at CMB Frequencies from Planck*, EAS Publications Series, Volume 23, (2007), pp.247-254
- [32] Marinucci, D., *High-Resolution Asymptotics for the Angular Bispectrum of Spherical Random Fields*, *The Annals of Statistics* 34, (2006) pp. 1-41
- [33] Marinucci, D., Pietrobon, D., Balbi, A., Baldi, P., Cabella, P., Kerkyacharian, G., Natoli, P., Picard, D., Vittorio, N. *Spherical Needlets for CMB Data Analysis*, *Monthly Notices of the Royal Astronomical Society*, Vol. 383, (2008), 539-545
- [34] Mayeli, A. *Asymptotic Uncorrelation for Mexican Needlets*, preprint (2008), arXiv:0806.3009
- [35] McEwen J. D., Vielva P., Hobson M. P., Martinez-Gonzalez E., Lasenby A. N. *Detection of the integrated Sachs-Wolfe effect and corresponding dark energy constraints made with directional spherical wavelets*, *Monthly Notices Roy. Astronom. Soc.*, 376 (3), (2007) 1211–1226.
- [36] McEwen J.D., Hobson M.P., Lasenby A.N., Mortlock, D.J. (2006) *A high-significance detection of non-Gaussianity in the WMAP 3-year data using directional spherical wavelets*, *Monthly Notices Roy. Astronom. Soc.*, 371, Issue 123002, (2006) L50–L54.
- [37] Narcowich, F.J. , Petrushev, P. and Ward, J. *Localized tight frames on spheres*, *SIAM J. Math. Anal.* **38** (2006), 574-594.
- [38] Narcowich, F.J., Petrushev, P. and Ward, J. *Decomposition of Besov and Triebel-Lizorkin spaces on the sphere*, *J. Func. Anal.* **238** (2006), 530-564.
- [39] Newman, E. T. and Penrose, R. *Notes on the Bondi-Metzner-Sachs Group*, *J. Math. Phys* **7** (1966) 863-870.
- [40] Parthasarathy, K.R. *Probability Measures on Metric Spaces*, (2005), AMS Chelsea Publishing (Reprint of the 1967 original)

- [41] Pietrobon, D., Balbi, A., Marinucci, D. *Integrated Sachs-Wolfe Effect from the Cross Correlation of WMAP3 Year and the NRAO VLA Sky Survey Data: New Results and Constraints on Dark Energy*, Physical Review D, 74, (2006), 043524
- [42] Pietrobon, D., Amblard, A., Balbi, A., Cabella, P., Cooray, A., Marinucci, D. *Needlet detection of features in WMAP CMB sky and the impact on anisotropies and hemispherical asymmetries* Physical Review D **78** (2008) 10 3504, arXiv:0809.0010.
- [43] Schwartzman, A., Mascarenhas, W, and Taylor, J. *Inference for Eigenvalues and Eigenvectors of Gaussian Symmetric Matrices*, Annals of Statistics, in press (2008).
- [44] Starck J.-L., Moudden Y., Abrial P., Nguyen M. *Wavelets, ridgelets and curvelets on the sphere*, Astronomy and Astrophysics, Volume 446, (2006) pp.1191–1204
- [45] Varshalovich, D.A., Moskalev, A.N. and Khersonskii, V.K. *Quantum Theory of Angular Momentum*, (1988), World Scientific, Singapore
- [46] Vielva P., Martinez-Gonzalez E., Barreiro B., Sanz J., Cayon L. *Detection of non-Gaussianity in the WMAP first year data using spherical wavelets*, Astrophysical J., Volume 609, (2004) pp. 22-34.
- [47] Vielva P., Wiaux I., Martinez-Gonzalez E., Vandergheynst P., *Alignment and signed-intensity anomalies in WMAP data*, Mon. Not. R. Astron. Soc. 381, (2007) 932–942
- [48] Vilenkin, N.Ja an Klimyk, A.U. *Representations of Lie Groups and Special Functions*, (1991), Kluwer Academic Press
- [49] Wiaux, Y., Jacques, L., Vandergheynst, P. *Correspondence Principle between Spherical and Euclidean Wavelets*, The Astrophysical Journal, Volume 632, Issue 1, (2005) pp. 15-28
- [50] Wiaux, Y., Jacques, L., Vandergheynst, P. *Fast spin ± 2 spherical harmonics and applications in cosmology*, Journal of Computational Physics, v. 226, iss. 2, (2007) p. 2359-2371
- [51] Wiaux, Y., McEwen, J.D., Vielva, P., *Complex Data Processing: Fast Wavelet Analysis on the Sphere*, Journal of Fourier Analysis and its Applications, 13, (2007) 477-494
- [52] Wiaux I., McEwen J. D., Vandergheynst P., Blanc O. *Exact reconstruction with directional wavelets on the sphere*, Monthly Notices of the Royal Astronomical Society, Volume 388, Issue 2, (2008) pp. 770-788.
- [53] Zaldarriaga, M. and Seljak, U. *All-sky analysis of polarization in the microwave background*, Phys. Rev. D **33** (1999) 1831-1840.